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INDEX.

ERRATA.

Page 54, line 7. For *collect-groups* read *generator-groups*.

" 106, § 31. The first formula should read $(A \pm B)\, C = AC \pm BC$.

" 126. The third formula should read $k\,(i - k) = j$.

" 132. Foot-note, second line of second paragraph, read $j = \frac{1}{2}(j_1 - \mathrm{J}k_1)$, $l = \frac{1}{2}(1 + \mathrm{J}i_1)$.

" 136. Last line of foot-note. For e, read l.

" 148. Multiplication table of (l_5). For $ji = i$, read $ji = j$.

" 171. Last line of foot-note, insert l, at beginning of line.

" 182. Foot-note. Add that on substituting $k + \mathbf{r}j$ for k, the algebra (aw_5) reduces to (ax_5), and the same substitution reduces (ay_5) to (az_5).

" 187. Last line of foot-note. For i, read l.

" 246, line 14. After the word *that* insert *with a value of h capable of being made less than any assignable quantity.*

On the 34 Concomitants of the Ternary Cubic.

By Professor Cayley, *Cambridge, England.*

I have (by aid of Gundelfinger's formulæ, afterwards referred to,) calculated, and I give in the present paper, the expressions of the 34 concomitants of the canonical ternary cubic $ax^3 + by^3 + cz^3 + 6lxyz$, or, what is the same thing, the 34 covariants of this cubic and the adjoint linear function $\xi x + \eta y + \zeta z$: this is the chief object of the paper. I prefix a list of memoirs, with short remarks upon some of them; and, after a few observations, proceed to the expressions for the 34 concomitants; and, in conclusion, exhibit the process of calculation of these concomitants other than such of them as are taken to be known forms. I insert a supplemental table of 6 derived forms.

The list of memoirs (not by any means a complete one) is as follows:

Hesse, Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln: *Crelle*, t. xxviii (1844), pp. 68–96. Although purporting to relate to a different subject, this is in fact the earliest, and a very important, memoir in regard to the general ternary cubic; and in it is established the canonical form, as Hesse writes it, $y_1^3 + y_2^3 + y_3^3 + 6\pi y_1 y_2 y_3$.

Aronhold, Zur Theorie der homogenen Functionen dritten Grades von drei Variabeln: *Crelle*, t. xxxix (1850), pp. 140–159.

Cayley, A Third Memoir on Quantics: *Phil. Trans.*, t. cxlvi (1856), pp. 627–647.

Aronhold, Theorie der homogenen Functionen dritten Grades von drei Variabeln: *Crelle*, t. lv (1858), pp. 97–191.

Salmon, Lessons Introductory to the Modern Higher Algebra: 8°, Dublin, 1859.

Cayley, A Seventh Memoir on Quantics: *Phil. Trans.*, t. cli (1861), pp. 277–292.

Brioschi, Sur la theorie des formes cubiques a trois indeterminées: *Comptes Rendus*, t. lvi (1863), pp. 304–307.

HERMITE, Extrait d'une lettre à M. Brioschi: *Crelle*, t. lxiii (1864), pp. 30–32, followed by a note by BRIOSCHI, pp. 32–33.

The skew covariant of the ninth order $(y^3 - z^3 \,.\, z^3 - x^3 \,.\, x^3 - y^3$ for the canonical form $x^3 + y^3 + z^3 + 6lxyz)$, and the corresponding contravariant $\eta^3 - \zeta^3 \,.\, \zeta^3 - \xi^3 \,.\, \xi^3 - \eta^3$, alluded to p. 116 of Salmon's Lessons, were obtained, the covariant by Brioschi and the contravariant by Hermite, in the last-mentioned papers.

CLEBSCH and GORDAN, Ueber die Theorie der ternären cubischen Formen: *Math. Annalen*, t. i (1869), pp. 56–89.

The establishment of the complete system of the 34 covariants, contravariants and *Zwischenformen*, or, as I have here called them, the 34 concomitants, was first effected by Gordan in the next following memoir:

GORDAN, Ueber die ternären Formen dritten Grades: *Math. Annalen*, t. i (1869), pp. 90–128.

And the theory is further considered:

GUNDELFINGER, Zur Theorie der ternären cubischen Formen: *Math. Annalen* t. vi (1871), pp. 144–163. The author speaks of the 34 forms as being "theils mit den von Gordan gewählten identisch, theils möglichst einfache Combinationen derselben." They are, in fact, the 34 forms given in the present paper for the canonical form of the cubic, and the meaning of the adopted combinations of Gordan's forms will presently clearly appear.

There is an advantage in using the form $ax^3 + by^3 + cz^3 + 6lxyz$ rather than the Hessian form $x^3 + y^3 + z^3 + 6lxyz$, employed in my Third and Seventh Memoirs on Quantics: for the form $ax^3 + by^3 + cz^3 + 6lxyz$ is what the general cubic $(a, b, c, f, g, h, i, j, k, l)(x, y, z)^3$ becomes by no other change than the reduction to zero of certain of its coefficients; and thus any concomitant of the canonical form consists of terms which are leading terms of the same concomitant of the general form.

The concomitants are functions of the coefficients $(a, b, \ldots l)$, of (ξ, η, ζ), and of (x, y, z): the dimensions in regard to the three sets respectively may be distinguished as the degree, class, and order; and we have thus to consider the deg-class-order of a concomitant.

Two or more concomitants of the same deg-class-order may be linearly combined together: viz., the linear combination is the sum of the concomitants each multiplied by a mere number. The question thus arises as to the selection of a representative concomitant. As already mentioned, I follow Gundelfinger,

viz., my 34 concomitants of the canonical form correspond each to each (with only the difference of a numerical factor of the entire concomitant) to his 34 concomitants of the general form. The principle underlying the selection would, in regard to the general form, have to be explained altogether differently; but this principle exhibits itself in a very remarkable manner in regard to the canonical form $ax^3 + by^3 + cz^3 + 6lxyz$.

Each concomitant of the general form is an indecomposable function, not breaking up into rational factors; but this is not of necessity the case in regard to a canonical form (only a concomitant which *does* break up must be regarded as indecomposable, no factor of such concomitant being rejected, or separated). So far from it, there is, in regard to the canonical form in question, a frequent occurrence of $abc + 8l^3$ or a power thereof, either as a factor of a unique concomitant, or when there are two or more concomitants of the same deg-class-order, then as a factor of a properly selected linear combination of such concomitants: and the principle referred to is in fact that of the selection of such combination for the representative concomitant; or (in other words) the representative concomitant is taken so as to contain as a factor the highest power that may be of $abc + 8l^3$. (As to the signification of this expression $abc + 8l^3$, I call to mind that the discriminant of the form is $abc(abc + 8l^3)^3$).

As to numerical factor: my principle has been, and is, to throw out any common numerical divisor of all the terms: thus I write $S = -abcl + l^4$, instead of Aronhold's $S = -4abcl + 4l^4$. There is also the question of nomenclature: I retain that of my Seventh Memoir on Quantics, except that I use single letters H, P, &c., instead of the same letters with U, thus HU, PU, &c.; in particular I use U, H, P, Q instead of Aronhold's f, Δ, S_f, T_f. It is thus at all events necessary to make some change in Gundelfinger's letters; and there is moreover a laxity in his use of accented letters; his B, B', B'', B''', and so in other cases E, E', E'', &c., are used to denote functions derived in a determinate manner each from the preceding one (by the δ-process explained further on); whereas his L, L'; M, M'; N, N' are functions having to each other an altogether different relation; also three of his functions are not denoted by any letters at all. Under the circumstances, I retain only a few of his letters; use the accent where it denotes the δ-process; and introduce barred letters $\overline{J}$, $\overline{K}$, &c., to denote a different correspondence with the unbarred letters J, K, &c. But I attach also to each concomitant a numerical symbol showing its deg-class-order, thus: 541 (degree $= 5$, class $= 4$, order $= 1$) or 1290, (there is no ambiguity in the two-digit numbers 10, 11, 12 which present themselves in the system

of the 34 symbols); and it seems to me very desirable that the significations of these deg-class-order symbols should be considered as permanent and unalterable. Thus, in writing $S = 400 = -abcl + l^4$, I wish the 400 to be regarded as denoting its expressed value $-abcl + l^4$: if the same letter S is to be used in Aronhold's sense to denote $-4abcl + 4l^4$, this would be completely expressed by the new definition $S = \mathbf{4}.400$, the meaning of the symbol 400 being explained by reference to the present memoir, or by the actual quotation $400 = -abcl + l^4$.

I proceed at once to the table: for shortness, I omit in general terms which can be derived from an expressed term by mere cyclical interchanges of the letters (a, b, c), (ξ, η, ζ), (x, y, z).

Table of the 34 Covariants of the Canonical Cubic $ax^3 + by^3 + cz^3 + 6lxyz$ and linear form $\xi x + \eta y + \zeta z$.

First Part, 10 Forms. Class = Order.

Current No.

1 $\quad S = 400 = -abcl + l^4.$

2 $\quad T = 600 = a^2b^2c^2 - 20abcl - 8l^6.$

3 $\quad \Lambda = 011 = \xi x + \eta y + \zeta z.$

4 $\quad \Theta = 222 = x^2[-l^2\xi^2 - 2al\eta\zeta] \,..$

$\qquad + yz[bc\xi^2 + 2l^2\eta\zeta] \,..$

5 $\quad \Theta' = 422 = x^2[l(abc + 2l^3)\xi^2 + a(abc - 4l^3)\eta\zeta] \,..$

$\qquad + yz[bcl^2\xi^2 - 2(abc + 2l^3)\eta\zeta] \,..$

6 $\quad \Theta'' = 622 = x^2[-(abc + 2l^3)^2\xi^2 + 12al^2(abc + 2l^3)\eta\zeta] \,..$

$\qquad + yz[36bcl^4\xi^2 + 2(abc + 2l^3)^2\eta\zeta] \,..$

7 $\quad B = 333 = x^3[a^2(c\eta^3 - b\zeta^3)] \,..$

$\qquad + y^2z[(abc + 8l^3)\eta^2\zeta + 12bl^2\zeta^2\xi + 6bcl\xi^2\eta] \,..$

$\qquad + yz^2[-(abc + 8l^3)\eta\zeta^2 - 6bcl\zeta\xi^2 - 12cl^2\xi\eta^2] \,..$

8 $\quad B' = 533 = x^3[3a^2l^2(c\eta^3 - b\zeta^3)] \,..$

$\qquad + y^2z[-l^2(abc + 8l^3)\eta^2\zeta + 4bl(-abc + l^3)\zeta^2\xi$

$\qquad\qquad - bc(abc - 10l^3)\xi^2\eta] \,..$

$\qquad + yz^2[l^2(abc + 8l^3)\eta\zeta^2 + bc(abc - 10l^3)\zeta\xi^2$

$\qquad\qquad - 4cl(-abc + l^3)\xi\eta^2] \,..$

9 $\quad B'' = 733 = x^3[9a^2l^4(c\eta^3 - b\zeta^3)] \,..$

$\qquad + y^2z[l(abc + 8l^3)(2abc + l^3)\eta^2\zeta$

$\qquad\qquad + b(abc + 2l^3)(abc - 10l^3)\zeta^2\xi + 6bcl^2(-abc + l^3)\xi^2\eta] \,..$

$\qquad + yz^2[-l(abc + 8l^3)(2abc + l^3)\eta\zeta^2$

$\qquad\qquad - 6bcl^2(-abc + l^3)\zeta\xi^2 - c(abc + 2l^3)(abc - 10l^3)\xi\eta^2] \,..$

Current No.

10 $B''' = 933 = \quad x^3\,[27a^2l^6\,(c\eta^3 - b\zeta^3)]\,\ldots$

$+\,y^2z\,[-\,(abc+8l^3)(abc-l^3)^2\eta^2\zeta + 9bl^2\,(abc+2l^3)^2\zeta^2\xi$

$-\,27bcl^4\,(abc+2l^3)\,\xi^2\eta]\,\ldots$

$+\,yz^2\,[(abc+8l^3)(abc-l^3)^2\eta\zeta^2 + 27bcl^4\,(abc+2l^3)\,\zeta\xi^2$

$-\,9cl^2\,(abc+2l^3)^2\xi\eta^2]\,\ldots$

Second Part, (4 + 4 =) 8 forms. Class = 0, and Order = 0:

Class = 0.

11 $U \;=\; 103 = ax^3 + by^3 + cz^3 + 6lxyz.$

12 $H \;=\; 303 = l^2\,(ax^3 + by^3 + cz^3) - (abc + 2l^3)\,xyz.$

13 $\Psi \;=\; 806 = (abc + 8l^3)^2\{a^2x^6 + b^2y^6 + c^2z^6 - 10\,(bcy^3z^3 + caz^3x^3 + abx^3y^3)\}.$

14 $\Omega \;=1209 = (abc + 8l^3)^3\{by^3 - cz^3.\,cz^3 - ax^3.\,ax^3 - by^3\}.$

Order = 0.

15 $P \;=\; 330 = -\,l\,(bc\xi^3 + ca\eta^3 + ab\zeta^3) + (-\,abc + 4l^3)\,\xi\eta\zeta.$

16 $Q \;=\; 530 = (abc - 10l^3)(bc\xi^3 + ca\eta^3 + ab\zeta^3) - 6l^2\,(5abc + 4l^3)\,\xi\eta\zeta.$

17 $F \;=\; 460 = b^2c^2\xi^6 + c^2a^2\eta^6 + a^2b^2\zeta^6 - 2\,(abc + 16l^3)(a\eta^3\zeta^3 + b\zeta^3\xi^3 + c\xi^3\eta^3)$

$-\,24l^3\,(bc\xi^3 + ca\eta^3 + ab\zeta^3)\,\xi\eta\zeta - 24l\,(abc + 2l^3)\,\xi^2\eta^2\zeta^2.$

18 $\Pi \;=1290 = (abc + 8l^3)^3\{c\eta^3 - b\zeta^3.\,a\zeta^3 - c\xi^3.\,b\xi^3 - a\eta^3\}.$

Third Part, (8 + 8 =) 16 forms. Class less or greater than Order.

Class less than Order.

19 $J \;=\; 414 = (abc + 8l^3)\{\,\xi x(by^3 - cz^3) + \eta y\,(cz^3 - ax^3) + \zeta z\,(ax^3 - by^3)\}.$

20 $K \;=\; 514 = (abc + 8l^3)\{\,\xi\,[alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2]\,\ldots\}.$

21 $K' \;=\; 714 = (abc + 8l^3)\{\,\xi\,[(abc + 2l^3)\,(ax^4 - 2blxy^3 - 2clxz^3)$

$-\,18bcl^2y^2z^2]\,\ldots\}.$

22 $E \;=\; 625 = (abc + 8l^3)\,\{\xi^2\,(by^3 - cz^3)\,[2l^2x^2 + bcyz]\,\ldots$

$+\,\eta\zeta\,(by^3 - cz^3)[4alx^2 + 2l^2yz]\,\ldots\}.$

23 $E' \;=\; 825 = (abc + 8l^3)\,\{\xi^2\,(by^3 - cz^3)[l\,(abc + 2l^3)\,x^2 - 3bcl^2yz]\,\ldots$

$+\,\eta\zeta\,(by^3 - cz^3)[\,a\,(abc - 4l^3)\,x^2 + l\,(abc + 2l^3)\,yz]\,\ldots\}.$

24 $E'' \;=1025 = (abc + 8l^3)\,\{\xi^2\,(by^3 - cz^3)[(abc + 2l^3)^2\,x^2 + 18bcl^4yz]\,\ldots$

$+\,\eta\zeta(by^3 - cz^3)[-12al^3\,(abc + 2l^3)\,x^2 + (abc + 2l^3)^2\,yz]\,\ldots\}.$

25 $M \;=\; 917 = (abc + 8l^3)^2\{\xi\,(by^3 - cz^3)[5alx^4 - blxy^3 - clxz^3 - 3bcy^2z^2]\,\ldots\}.$

26 $M' = 1117 = (abc + 8l^3)^2\{\xi\,(by^3 - cz^3)[(abc + 2l^3)(5ax^4 - bxy^3 - cxz^3)$

$+\,18bcl^2y^2z^2]\,\ldots\}.$

Order less than Class.

27 $\bar{J} \;=\; 841 = (abc + 8l^3)^2\{x\xi a(c\eta^3 - b\zeta^3) + y\eta b(a\zeta^3 - c\xi^3) + z\zeta c(b\xi^3 - a\eta^3)\}.$

28 $\bar{K} \;=\; 541 = (abc + 8l^3)\{\,x\,[bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3 - 6al\eta^2\zeta^2]\,\ldots\}.$

Current No. Order less than Class.

29 $\overline{K}' = 741 = (abc+8l^3)\{x[l^2(bc\xi^4-2ca\xi\eta^3-2ab\xi\zeta^3)+a(abc+2l^3)\eta^2\zeta^2]\ldots\}.$

30 $\overline{E} = 652 = (abc+8l^3)\{x^2(c\eta^3-b\zeta^3)[2al\xi^2+a^2\eta\zeta]\ldots$
$+ yz(c\eta^3-b\zeta^3)[4l^2\xi^2+2al\eta\zeta]\ldots\}.$

31 $\overline{E}' = 852 = (abc+8l^3)\{x^2(c\eta^3-b\zeta^3)[a(abc-4l^3)\xi^2-6a^2l^2\eta\zeta]\ldots$
$+ yz(c\eta^3-b\zeta^3)[4l(abc+2l^3)\xi^2+a(abc-4l^3)\eta\zeta]\ldots\}.$

32 $\overline{E}'' = 1052 = (abc+8l^3)\{x^2(c\eta^3-b\zeta^3)[-3al^2(abc+2l^3)\xi^2+9a^2l^4\eta\zeta]\ldots$
$+yz(c\eta^3-b\zeta^3)[(abc+2l^3)^2\xi^2-3al^2(abc-4l^3)\eta\zeta]\ldots\}.$

33 $\overline{M} = 771 = (abc+8l^3)\{x(c\eta^3-b\zeta^3)[(abc-8l^3)\xi^4-a^2c\xi\eta^3-a^2b\xi\zeta^3$
$-12al^2\xi^2\eta\zeta-6a^2l\eta^2\zeta^2]\ldots\}.$

34 $\overline{M}' = 971 = (abc+8l^3)\{x(c\eta^3-b\zeta^3)[l^2(7abc+8l^3)\xi^4-3a^2cl^2\xi\eta^3-3a^2bl^2\xi\zeta^3$
$+4al(abc-l^3)\xi^2\eta\zeta+a^2(abc-10l^3)\eta^2\zeta^2]\ldots\}.$

To this may be joined the following Supplemental Table of certain Derived Forms:

35. $R = 1200 = 64S^3-T^2 = -abc(abc+8l^3)^3.$

36 $C = 703 = -TU+24SH = (abc+8l^3)\{(-abc+4l^3)(ax^3+by^3+cz^3)$
$+18abclxyz\}.$

37 $D = 903 = 8S^2U-3TH = (abc+8l^3)\{l^2(5abc+4l^3)(ax^3+by^3+cz^3)$
$+3abc(abc-10l^3)xyz\}.$

38 $Y = 930 = 3TP-4SQ = (abc+8l^3)^2\{l(bc\xi^3+ca\eta^3+ab\zeta^3)$
$-3abc\xi\eta\zeta\}.$

39 $Z = 1130 = -48S^2P+TQ = (abc+8l^3)^2\{(abc+2l^3)(bc\xi^3+ca\eta^3+ab\zeta^3)$
$+18abcl^2\xi\eta\zeta\}.$

40 $\Phi = 1640 = 12(abc+8l^3)^2F-288STP^2+768S^2PQ-8TQ^2$
$= (abc+8l^3)^4\{b^2c^2\xi^6+c^2a^2\eta^6+a^2b^2\zeta^6$
$-10abc(a\eta^3\zeta^3+b\zeta^3\xi^3+c\xi^3\eta^3)\}.$

viz., these are derived forms characterized by having a power of $abc+8l^3$ as a factor: R is the discriminant; C, D, Y, Z occur in Aronhold, and see my Seventh memoir on Quantics: Φ in Clebsch and Gordan's memoir of 1869.

I regard as known forms Λ, U, H, P, Q, S, T, F, that is, the eight forms 3, 11, 12, 15, 16, 1, 2, 17; the remaining 26 forms are expressed in terms of these by formulæ involving notations which will be explained, viz: We have

13 $\Psi = 3(\mathrm{bc'+b'c-2ff'}, \ldots \mathrm{gh'+g'h-af'-a'f}, \ldots \between X, Y, Z \between X', Y', Z') + TU^2.$

14 $\Omega = \frac{1}{18}\,\mathrm{Jac}(U, H, \Psi).$

18 $\Pi \;=-\frac{1}{36}\,[\mathrm{Jac}]\,(P,\ Q,\ F).$
4 $\Theta \;=\quad (\mathrm{bc}-\mathrm{f}^2,\ ..\ \mathrm{gh}-\mathrm{af},\ ..\,)(\xi,\ \eta,\ \zeta)^2.$
5 $\Theta' =\quad \frac{1}{2}\,\delta\Theta\,.$
6 $\Theta' =\quad \frac{1}{2}\,\delta^2\Theta\,.$
7 $B \;=-\frac{1}{3}\,\mathrm{Jac}\,(U,\ \Theta,\ \Lambda)\,.$
8 $B' =\quad \frac{1}{6}\,\delta B\,.$
9 $B'' =\quad \frac{1}{24}\,\delta^2 B\,.$
10 $B''' =\quad \frac{1}{48}\,\delta^3 B\,.$
19 $J \;=-\frac{1}{3}\,\mathrm{Jac}\,(U,\ H,\ \Lambda)\,.$
27 $\overline{J} \;=\quad \frac{1}{3}\,[\mathrm{Jac}]\,(P,\ Q,\ \Lambda)\,.$
20 $K \;=-\frac{3}{2}\,\{\partial_\xi\Theta\partial_x H+\partial_\eta\Theta\partial_y H+\partial_\zeta\Theta\partial_z H\}-SU\Lambda\,.$
21 $K' =-(\delta)\,K,$
28 $\overline{K} =\quad 3\,\{\partial_x\Theta\partial_\xi P+\partial_y\Theta\partial_\eta P+\partial_z\Theta\partial_\zeta P\}+\quad Q\Lambda\,.$
29 $\overline{K}' =\quad \frac{1}{6}\,(\delta)\,\overline{K}.$
22 $E \;=-\frac{1}{18}\,\mathrm{Jac}\,(K,\ U,\ \Lambda)\,.$
23 $E' =-\frac{1}{4}\,(\delta)\,E.$
24 $E'' =\quad \frac{1}{4}\,(\delta^2)\,E.$
30 $\overline{E} \;=-\frac{1}{9}\,\mathrm{Jac}\,(\overline{K},\ U,\ \Lambda)\,.$
31 $\overline{E}' =-\frac{1}{2}\,(\delta)\,\overline{E}.$
32 $\overline{E}'' =-\frac{1}{8}\,(\delta^2)\,\overline{E}.$
25 $M \;=\quad \frac{1}{36}\,\mathrm{Jac}\,(U,\ \Psi,\ \Lambda)\,.$
26 $M' =-(\delta)\,M.$
33 $\overline{M} \;=-\frac{1}{6}\,[\mathrm{Jac}](P,\ F,\ \Lambda)\,.$
34 $\overline{M}' =\quad \frac{1}{6}\,(\delta)\,\overline{M}.$

In explanation of the notations, observe that

$$U = \quad ax^3+by^3+cz^3\;+6lxyz,$$
$$H = l^3(ax^3+by^3+cz^3)-(abc\;+2l^3)\,xyz\,.$$

Hence, writing

$$6H = \quad a'x^3+b'y^3+c'z^3+6l'xyz,$$

we have

$$a',\ b',\ c',\ l' = 6al^2,\ 6bl^2,\ 6cl^2,\ -(abc\;+2l^3).$$

And this being so, we write

$$X,\ Y,\ Z \qquad = ax^2+2lyz,\ by^2+2lzx,\ cz^2+2lxy,$$
$$\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h} = ax,\ by,\ cz,\ lx,\ ly,\ lz,$$

for $\frac{1}{3}$ of the first differential coefficients, and $\frac{1}{6}$ of the second differential coefficients of U; and in like manner

$$\begin{aligned} X',\ Y',\ Z' &= a'x^2 + 2l'yz,\quad b'y^2 + 2l'zx,\quad c'z^2 + 2l'xy, \\ \mathrm{a}',\ \mathrm{b}',\ \mathrm{c}',\ \mathrm{f}',\ \mathrm{g}',\ \mathrm{h}' &= a'x,\ b'y,\ c'z,\ l'x,\ l'y,\ l'z, \end{aligned}$$

for $\frac{1}{3}$ of the first differential coefficients, and $\frac{1}{6}$ of the second differential coefficients of $6H$.

Jac. is written to denote the Jacobian, viz:

$$\mathrm{Jac}\,(U,\ H,\ \Psi) = \begin{vmatrix} \partial_x U, & \partial_y U, & \partial_z U \\ \partial_x H, & \partial_y H, & \partial_z H \\ \partial_x \Psi\,, & \partial_y \Psi\,, & \partial_z \Psi \end{vmatrix},$$

and in like manner [Jac] to denote the Jacobian, when the differentiations are in regard to $(\xi,\ \eta,\ \zeta)$ instead of $(x,\ y,\ z)$: δ is the symbol of the δ-process, or substitution of the coefficients $(a',\ b',\ c',\ l')$ in place of $(a,\ b,\ c,\ l)$; in fact $\delta = a'\partial_a + b'\partial_b + c'\partial_c + l'\partial_l$: δ, δ^2, &c., each operate directly on a function of $(a,\ b,\ c,\ l)$, the $(a',\ b',\ c',\ l')$ of the symbol δ being in the first instance regarded as constants, and being replaced ultimately by their values; for instance, $\delta abc = a'bc + ab'c + abc'$, $\delta^2 abc = 2\,(ab'c' + a'bc' + a'b'c)$, $\delta^3 abc = 6a'b'c'$.

In several of the formulæ, instead of δ or δ^2, the symbol used is (δ) or (δ^2); in these cases the function operated upon contains the factor $(abc + 8l^3)$ or $(abc + 8l^3)^2$, and is of the form $(abc + 8l^3)(aU + bV + cW)$ or $(abc + 8l^3)^2 (a^2U + abV + \&c.)$: the meaning is, that the δ or δ^2 is supposed to operate through the $(abc + 8l^3)a$, or $(abc + 8l^3)^2a^2$, &c., as if this were a constant, upon the U, V, &c., only; thus: $(\delta)\,.\,(abc + 8l^3)(aU + bV + cW)$ is used to denote $(abc + 8l^3)(a\delta U + b\delta V + c\delta W)$. As to this, observe that, operating with δ instead of (δ), there would be the additional terms $U\delta\,(abc + 8l^3)a$ + &c.; we have in this case $\delta\,(abc + 8l^3)a$, $= a\,(2a'bc + ab'c + abc' + 24l^2l') + 8l^3a'$, $= 24a^2bcl^2 - 24al^2\,(abc + 2l^3) + 48al^5$, $= 0$; or the rejected terms in fact vanish. For $(\delta^2)\,.\,(abc + 8l^3)\,(aU + bV + cW)$, operating with δ^2, we should have, in like manner, terms $U\delta^2\,(abc + 8l^3)a$, &c.; here $\delta^2\,(abc + 8l^3)a = a'^2bc + 2aba'c' + 2aca'b' + a^2b'c' + 24l^2a'l' + 24all'^2$, which is found to be $= -\,24a\,(abc + 8l^3)(-\,abcl + l^4)$, that is, $= -\,24S\,(abc + 8l^3)a$; and the terms in question are thus $= -\,24S\,(abc + 8l^3)\,(aU + bV + cW)$, viz. $(abc + 8l^3)\,(aU + bV + cW)$ being a covariant, this is also a covariant; that is, in using (δ^2) instead of δ^2, we in fact reject certain covariant terms; or say, for instance, δ^2E being a covariant, then $(\delta^2)E$ is also a covariant, but a different covariant. The calculation with (δ)

or (δ^3) is more simple than it would have been with δ or δ^2. See *post*, the calculations of K', $\overline{K}'$, &c.

I give for each of the 26 covariants a calculation showing how at least a single term of the final result is arrived at, and, in the several cases for which there is a power of $abc + 8l^3$ as a factor, showing how this factor presents itself.

Calculations for the 26 Covariants.

13. $\Psi = 3\,(\mathrm{bc'} + \mathrm{b'c} - 2\mathrm{ff'}, \ldots \mathrm{gh'} + \mathrm{g'h} - \mathrm{af'} - \mathrm{a'f}, \ldots \between X, Y, Z \between X', Y', Z') + TU^2,$

$$= 3\,((bc' + b'c)yz - 2ll'x^2, .\, 2ll'yz - (al' + a'l)x^2, . \between ax^2 + 2lyz, . \between a'x^2 + 2l'yz, .) + T(a^2x^6 + \ldots).$$

The whole coefficient of x^6 is

$$-6ll'aa' + Ta^2, \quad = 36a^2l^3\,(abc + 2l^3) + Ta^2,$$

viz. the coefficient of a^2x^6 is

$$\begin{aligned} &= 36l^3\,(abc + 2l^3) + a^2b^2c^2 - 20abcl^3 - 8l^6 \\ &= a^2b^2c^2 + 16abcl^3 + 64l^6, \\ &= (abc + 8l^3)^2. \end{aligned}$$

14. $$\Omega = \tfrac{1}{18}\,\mathrm{Jac}\,(U, H, \Psi), = \tfrac{1}{2} \begin{vmatrix} X, & X', & \tfrac{1}{6}\,\partial_x\Psi \\ Y, & Y', & \tfrac{1}{6}\,\partial_y\Psi \\ Z, & Z', & \tfrac{1}{6}\,\partial_z\Psi \end{vmatrix}.$$

Here

$$\begin{aligned} YZ' - Y'Z &= (by^2 + 2lzx)(c'z^2 + 2l'xy) - (cz^2 + 2lxy)(b'z^2 + 2l'xy), \\ &= (bc' - b'c)\,y^2z^2 + (2\,bl' - b'l)\,xy^3 - 2\,(cl' - c'l)\,xz^3, \\ &= -2\,(abc + 8l^3)\,x\,(by^3 - cz^3)\,; \end{aligned}$$

$$\tfrac{1}{2}\cdot\tfrac{1}{6}\,\partial_x\Psi = \tfrac{1}{2}\,(a^2x^5 - 5abx^2y^3 - 5acx^2z^3).$$

Hence the whole is

$$\begin{aligned} &= -(abc + 8l^3)\{a^2x^6\,(by^3 - cz^3) + b^2y^6\,(cz^3 - ax^3) + c^2z^6\,(ax^3 - by^3)\}, \\ &= \quad (abc + 8l^3)(by^3 - cz^3)(cz^3 - ax^3)(ax^3 - by^3).. \end{aligned}$$

18. $$\Pi = -\tfrac{1}{36}\,[\mathrm{Jac}](P, Q, F) = -\tfrac{1}{36} \begin{vmatrix} \partial_\xi P, & \partial_\xi Q, & \partial_\xi F \\ \partial_\eta P, & \partial_\eta Q, & \partial_\eta F \\ \partial_\zeta P, & \partial_\zeta Q, & \partial_\zeta F \end{vmatrix},$$

viz. if, in this calculation, we write

$6P = \mathrm{a}\xi^3 + \mathrm{b}\eta^3 + \mathrm{c}\zeta^3 + 6\mathrm{l}\xi\eta\zeta$, *i. e.* $\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{l} = -6lbc,\ -6lca,\ -6lab,\ -abc + 4l^3$

$Q = \mathrm{a}'\xi^3 + \mathrm{b}'\eta^3 + \mathrm{c}'\zeta^3 + 6\mathrm{l}'\xi\eta\zeta$, „ $\mathrm{a}', \mathrm{b}', \mathrm{c}', \mathrm{l}' = (abc - 10l^3)(bc,\ ca,\ ab),\ -l^2\,(5abc + 4l^3)$,

then

$$\Pi = -\tfrac{1}{4} \begin{vmatrix} \mathrm{a}\xi^2 + 2\,\mathrm{l}\eta\zeta\,, & \mathrm{a}'\xi^2 + 2\,\mathrm{l}'\eta\zeta\,, & \tfrac{1}{6}\,\partial_\xi F \\ \mathrm{b}\eta^2 + 2\,\mathrm{l}\zeta\xi\,, & \mathrm{b}'\eta^2 + 2\,\mathrm{l}'\zeta\xi\,, & \tfrac{1}{6}\,\partial_\eta F \\ \mathrm{c}\zeta^2 + 2\,\mathrm{l}\xi\eta\,, & \mathrm{c}'\zeta^2 + 2\,\mathrm{l}'\xi\eta\,, & \tfrac{1}{6}\,\partial_\zeta F \end{vmatrix}.$$

Here

$$(\mathrm{b}\eta^2 + 2\,\mathrm{l}\zeta\xi)(\mathrm{c}'\eta^2 + 2\,\mathrm{l}'\xi\eta) - (\mathrm{b}'\eta^2 + 2\,\mathrm{l}'\zeta\xi)(\mathrm{c}\eta^2 + 2\,\mathrm{l}\zeta\xi)$$
$$= (\mathrm{bc}' - \mathrm{b}'\mathrm{c})\,\eta^2\zeta^2 + 2\,(\mathrm{bl}' - \mathrm{b}'\mathrm{l})\,\xi\eta^3 - 2\,(\mathrm{cl}' - \mathrm{c}'\mathrm{l})\,\xi\zeta^3,$$

or since

$$\mathrm{bc}' - \mathrm{b}'\mathrm{c} = 0\,,$$
$$\mathrm{bl}' - \mathrm{b}'\mathrm{l} = -\,6lca\,.\,-\,l^2(5abc + 4l^3) - (abc - 10l^3)\,ca\,(-\,abc + 4l^3)$$
$$= ca\,\{6l^3(5abc + 4l^3) + (abc - 4l^3)(abc - 10l^3)\}$$
$$= ca\,(abc + 8l^3)^2,$$

and the like for $\mathrm{cl}' - \mathrm{c}'\mathrm{l}$, the expression is

$$= 2\,(abc + 8l^3)^2(ca\eta^3 - ab\zeta^3)\,\xi\,;$$

and the whole is thus

$$= -\tfrac{1}{2}\,(abc + 8l^3)^2\{(ca\eta^3 - ab\zeta^3)\,\xi\,.\,\tfrac{1}{6}\,\partial_\xi F + \,.\,.\}$$
$$= -\tfrac{1}{2}\,(abc + 8l^3)^2\{(ca\eta^3 - ab\zeta^3)[b^2c^2\xi^6 - (abc + 16l^3)(b\zeta^3\xi^3 + c\xi^3\eta^3) + \&\text{c.}]$$
$$+ (ab\zeta^3 - bc\xi^3)[c^2a^2\eta^6 - (abc + 16l^3)(c\xi^3\eta^3 + a\eta^3\zeta^3) + \&\text{c.}]$$
$$+ (bc\xi^3 - ca\eta^3)[a^2b^2\zeta^6 - (abc + 16l^3)(a\eta^3\zeta^3 + b\zeta^3\xi^3) + \&\text{c.}]\}.$$

Here the coefficient of $\xi^6\eta^3$, inside the $\{\}$, is

$$ab^2c^3 + bc^2\,(abc + 16l^3),\ = 2bc^2\,(abc + 8l^3),$$

and consequently the whole is

$$= -\,(abc + 8l^3)^3(bc^2\xi^6\eta^3 - \,.\,.),$$
$$= \quad (abc + 8l^3)^3\{(c\eta^3 - b\zeta^3)(a\zeta^3 - c\xi^3)(b\xi^3 - a\eta^3)\}.$$

4. $$\Theta = \quad (\mathrm{bc} - \mathrm{f}^2,\,.\,.\,\mathrm{gh} - \mathrm{af},\,.\,.\between\xi,\,\eta,\,\zeta)^2$$
$$= \quad (bcyz - l^2x^2)\xi^2\,.\,.\, + 2\,(l^2yz - alx^2)\eta\zeta\,.\,.$$

which are the terms of the final result

$$\Theta = x^2\,[-\,l^2\xi^2 - 2al\eta\zeta] + yz\,[bc\xi^2 + 2l^2\eta\zeta].$$

5 and 6. The δ-process applied to the terms of Θ just written down gives

$$\Theta' = \tfrac{1}{2}\,\delta\Theta = x^2\,[ll'\xi^2 - (al' + a'l)\,\eta\zeta] + yz\,[\tfrac{1}{2}(bc' + b'c)\,\xi^2 + 2ll'\eta\zeta],$$
$$\Theta'' = \tfrac{1}{2}\delta^2\Theta = x^2\,[-\,l'^2\xi^2 - 2a'l'\eta\zeta] + yz\,[b'c'\xi^2 + 2l'^2\eta\zeta],$$

and substituting for a', b', c', l' their values, we have the corresponding terms of Θ' and Θ'' respectively.

7. $$B = -\,\tfrac{1}{3}\,\mathrm{Jac.}(U,\,\Theta,\,\Lambda),\ = -\begin{vmatrix} X, & \partial_x\Theta, & \xi \\ Y, & \partial_y\Theta, & \eta \\ Z, & \partial_z\Theta, & \zeta \end{vmatrix}.$$

A term is $X(\eta\partial_z\Theta - \zeta\partial_y\Theta)$, and if, in this calculation, we write $\Theta = (A,\,B,\,C,\,F,\,G,\,H\between x,\,y,\,z)^2$, *i.e.* $A = -\,l^2\xi^2 - 2al\eta\zeta,\,.\,.\,F = \tfrac{1}{2}\,bc\xi^2 + l^2\eta\zeta,$ then the term is

$$= (ax^2 + 2lyz)\{x\,.\,2(G\eta - H\zeta) + y\,.\,2(F\eta - B\zeta) + z\,.\,2(C\eta - F\zeta)\}.$$

Here

$$2\,(G\eta - H\zeta) = \eta(ca\eta^2 + l^2\zeta\xi) - \zeta(ab\zeta^2 + l^2\xi\eta), = a(c\eta^3 - b\zeta^3),$$

and hence the whole term in x^3 is $= a^2x^3(c\eta^3 - b\zeta^3)$.

8, 9, 10. The coefficient of $x^3\eta^3$ in B is a^2c, and hence in δB, $\delta^2 B$, $\delta^3 B$ the coefficients of this term are $2a'ac + a^2c'$, $2a'^2c + 4aa'c'$, $6a'^2c'$, whence in B', B'', $B''' = \frac{1}{6}\,\delta B$, $\frac{1}{24}\,\delta^2 B$, $\frac{1}{48}\,\delta^3 B$ respectively, the coefficients are

$$\tfrac{1}{6}\,(a^2c' + 2aa'c), \quad \tfrac{1}{12}\,(a'^2c + 2aa'c'), \quad \tfrac{1}{8}\,a'^2c',$$

$$= \quad 3l^2a^2c, \qquad 9l^4a^2c, \qquad 27l^6a^2c \text{ respectively.}$$

19. $$J = -\tfrac{1}{3}\,\text{Jac}\,(U,\,H,\,\Lambda) = -\tfrac{1}{2}\begin{vmatrix} X, & X', & \xi \\ Y, & Y', & \eta \\ Z, & Z', & \zeta \end{vmatrix};$$

a term is $-\frac{1}{2}\,(YZ' - Y'Z)\,\xi$, where, as in a previous calculation,

$$YZ' - Y'Z = -2\,(abc + 8l^3)\,x\,(by^3 - cz^3).$$

Hence, whole is

$$= (abc + 8l^3)\{\xi x\,(by^3 - cz^3) + \eta y\,(cz^3 - ax^3) + \zeta z\,(ax^3 - by^3)\}.$$

27. $$\bar{J} = \tfrac{1}{3}\,[\text{Jac}]\,(P,\,Q,\,\Lambda) = \tfrac{1}{2}\begin{vmatrix} \mathrm{a}\xi^2 + 2\,\mathrm{l}\eta\zeta, & \mathrm{a}'\xi^2 + 2\,\mathrm{l}'\eta\zeta, & x \\ \mathrm{b}\eta^2 + 2\,\mathrm{l}\zeta\xi, & \mathrm{b}'\eta^2 + 2\,\mathrm{l}'\zeta\xi, & y \\ \mathrm{c}\zeta^2 + 2\,\mathrm{l}\xi\eta, & \mathrm{c}'\zeta^2 + 2\,\mathrm{l}'\xi\eta, & z \end{vmatrix},$$

if, as in a previous calculation

$$6P = \mathrm{a}\xi^3 + \mathrm{b}\eta^3 + \mathrm{c}\zeta^3 + 6\,\mathrm{l}\xi\eta\zeta, \quad Q = \mathrm{a}'\xi^3 + \mathrm{b}'\eta^3 + \mathrm{c}'\zeta^3 + 6\,\mathrm{l}'\xi\eta\zeta.$$

Here, as before,

$$(\mathrm{b}\eta^2 + 2\,\mathrm{l}\zeta\xi)(\mathrm{c}'\zeta^2 + 2\,\mathrm{l}'\xi\eta) - (\mathrm{b}'\eta^2 + 2\,\mathrm{l}'\zeta\xi)(\mathrm{c}\zeta^2 + 2\,\mathrm{l}\xi\eta) = 2\,(abc + 8l^3)^2(ca\eta^3 - ab\zeta^3)\xi.$$

Hence, whole is

$$= (abc + 8l^3)^2\{x\xi a\,(c\eta^3 - b\zeta^3) + y\eta b\,(a\zeta^3 - c\xi^3) + z\zeta c\,(b\xi^3 - a\eta^3)\}.$$

20. $K = -\frac{8}{3}\,(\partial_\xi\Theta\partial_x H + \partial_\eta\Theta\partial_y H + \partial_\zeta\Theta\partial_z H) - SU\Lambda$,
which, H being

$$= \tfrac{1}{6}\,(a'x^3 + b'y^3 + c'z^3 + 6l'xyz),$$

and putting $\Theta = (A,\,B,\,C,\,F,\,G,\,H\between\xi,\,\eta,\,\zeta)^2$, $A = -l^2x^2 + bcyz, \ldots$ $F = -alx^2 + l^2yz, \ldots$ is

$$\begin{aligned} = -\tfrac{8}{3}\,\{&(a'x^2 + 2l'yz)(A\xi + H\eta + G\zeta) \qquad - (-abcl + l^4)\,U\,(\xi x + \eta y + \zeta z) \\ +\,&(b'y^2 + 2l'zx)(H\xi + B\eta + F\zeta) \\ +\,&(c'z^2 + 2l'xy)(G\xi + F\eta + C\zeta)\}. \end{aligned}$$

The whole coefficient of ξ is thus

$$\begin{aligned}
&= -\tfrac{3}{2}\{(a'x^2 + 2l'yz)\,A + (b'y^2 + 2l'zx)\,H + (c'z^2 + 2l'xy)\,G\} - (-abcl + l^4)\,Ux \\
&= -\tfrac{3}{2}\{(a'x^2 + 2l'yz)(-l^2x^2 + bcyz) + (b'y^2 + 2l'zx)(-clz^2 + l^2xy) \\
&\qquad + (c'z^2 + 2l'xy)(-bly^2 + l^2zx)\} - (-abcl + l^4)\{ax^4 + bxy^3 + cxz^3 + 6lx^2yz\},
\end{aligned}$$

and herein the coefficient of x^4 is

$$= \tfrac{3}{2}a'l^2 - al\,(-abc + l^3), \quad = 9al^4 - al\,(-abc + l^3), \quad = (abc + 8l^3)\,al\,;$$

viz. we have thus the term $(abc + 8l^3)\,\xi\,.\,alx^4$ of the final result.

21. $K' = -(\delta)\,K$, where K is of the form $(abc + 8l^3)\,(aU + bV + cW)$, and operating with (δ), we obtain $(abc + 8l^3)(a\delta U + b\delta V + c\delta W)$. Taking for instance the term of K, $(abc + 8l^3)\,\xi\,[alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2]$, then, in operating with (δ), the term bc may be considered indifferently as belonging to bV or cW, and the resulting term of K' is

$$\begin{aligned}
K' = -(\delta)\,K &= -(abc + 8l^3)\,\xi\,[al'x^4 - 2bl'xy^3 - 2cl'xz^3 + 3bc'y^2z^2], \\
&= (abc + 8l^3)\,\xi\,[(abc + 2l^3)(ax^4 - 2bxy^3 - 2cxz^3) - 18bcl^2y^2z^2]
\end{aligned}$$

28. $\overline{K} = 3\{\partial_x\Theta\partial_\xi P + \partial_y\Theta\partial_\eta P + \partial_z\Theta\partial_\zeta P\} + Q\Lambda$; viz. writing $\Theta = (A, B, C, F, G, H \between x, y, z)^2$, $\quad A = -l^2\xi^2 - 2al\eta\zeta, \ldots F = \tfrac{1}{2}\,bc\xi^2 + l^2\eta\zeta, \ldots$ then this is

$$\begin{aligned}
= 3\{&[-3bcl\xi^2 + (-abc + 4l^3)\,\eta\zeta]\;2\,(Ax + Hy + Gz) \\
&+ [-3cal\eta^2 + (-abc + 4l^3)\,\zeta\xi]\;2\,(Hx + By + Fz) \\
&+ [-3abl\zeta^2 + (-abc + 4l^3)\,\xi\eta]\;2\,(Gx + Fy + Cz)\} \\
&+ \{(abc - 10l^3)(bc\xi^3 + ca\eta^3 + ab\zeta^3) - 6l^2\,(5abc + 4l^3)\,\xi\eta\zeta\}(\xi x + \eta y + \zeta z).
\end{aligned}$$

The whole coefficient of x is thus

$$\begin{aligned}
= 3\{&[-3bcl\xi^2 + (-abc + 4l^3)\,\eta\zeta]\,(-2l^2\xi^2 - 4al\eta\zeta) \\
&+ [-3cal\eta^2 + (-abc + 4l^3)\,\zeta\xi](ab\zeta^2 + l^2\xi\eta) \\
&+ [-3abl\zeta^2 + (-abc + 4l^3)\,\xi\eta](ac\eta^2 + l^2\zeta\xi)\} \\
&+ \{(abc - 10l^3)(bc\xi^4 + ca\xi\eta^3 + ab\xi\zeta^3) - 6l^2\,(5abc + 4l^3)\,\xi^2\eta\zeta\},
\end{aligned}$$

and herein the coefficient of ξ^4 is $18bcl^3 + (abc - 10l^3)\,bc$, $= (abc + 8l^3)\,bc$, giving, in the final result, the term $(abc + 8l^3)\,\xi\,.\,bcx^4$.

29. $$\overline{K'} = \tfrac{1}{6}\,(\delta)\,\overline{K}.$$

Here $\overline{K}$ is of the form $(abc + 8l^3)(aU + bV + cW)$, and we have

$$\overline{K'} = \tfrac{1}{6}\,(abc + 8l^3)(a\delta U + b\delta V + c\delta W).$$

A term of $aU + bV + cW$ is $x\,[bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3 - 6al\eta^2\zeta^2]$ where $bc\xi^4$ may be considered as belonging indifferently to bV or cW, and so for the other terms. The resulting term in $\frac{1}{6}(a\delta U + b\delta V + c\delta W)$ is thus

$$\tfrac{1}{6}\,x\,[bc'\xi^4 - 2ca'\xi\eta^3 - 2ab'\xi\zeta^3 - 6al'\eta^2\zeta^2],$$

which is $= x\,[l^2(bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3) + a(abc + 2l^3)\,\eta^2\zeta^2]$, and we have thus a term of $\overline{K}'$.

22. $$E = -\tfrac{1}{18}\,\text{Jac}\,(K,\ U,\ \Lambda):$$

K contains the factor $abc + 8l^3$, and if, omitting this factor, the value of K is called $A\xi + B\eta + C\zeta$, then we have

$$\begin{aligned} E = -\tfrac{1}{6}\,\{&(\xi\partial_x A + \eta\partial_x B + \zeta\partial_x C)(Y\zeta - Z\eta) + (\xi\partial_y A + \eta\partial_y B + \zeta\partial_y C)(Z\xi - X\zeta) \\ &+ (\xi\partial_z A + \eta\partial_z B + \zeta\partial_z C)(X\eta - Y\xi)\}, \end{aligned}$$

and the term herein in ξ^2 is $-\frac{1}{6}\,\xi^2\,(Z\partial_y A - Y\partial_z A)$, where A is

$$= alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2;\ \text{viz. the coefficient of } \xi^2 \text{ is}$$

$$\begin{aligned} &= -\tfrac{1}{6}\,\{(cz^2 + 2lxy)(-6blxy^2 + 6bcyz^2) - (by^2 + 2lzx)(-6clxz^2 + 6bcy^2z)\}, \\ &= \quad b^2cy^4z - bc^2yz^4 + 2bl^2x^2y^3 - 2cl^2x^2z^3, \\ &= \quad (2l^2x^2 + bcyz)\,(by^3 - cz^3): \end{aligned}$$

Hence, restoring the omitted factor $(abc + 8l^3)$, we have in E the term

$$(abc + 8l^3)\,\xi^2\,(by^3 - cz^3)\,[2l^2x^2 + bcyz].$$

23, 24. $$E' = -\tfrac{1}{4}\,(\delta)\,E;\quad E'' = \tfrac{1}{4}\,(\delta^2)\,E:$$

E is of the form $(abc + 8l^3)\,(aU + bV + cW)$, and, as before, in a term such as $(abc + 8l^3)\,\xi^2\,(by^3 - cz^3)\,(2l^2x^2 + bcyz)$ we operate with δ or δ^2 only on the factor $2l^2x^2 + bcyz$; and in E' and E'' respectively, operating upon this factor, we obtain

$$-\tfrac{1}{4}\,\{4ll'x^2 + (bc' + b'c)\,yz\},\quad \text{and}\quad \tfrac{1}{4}\,\{4l'^2x^2 + 2b'c'yz\},$$

viz. we thus obtain in E' the term

$$(abc + 8l^3)\,\xi^2\,(by^3 - cz^3)\,[l\,(abc + 2l^3)\,x^2 - 3bcl^2yz],$$

and in E'' the term

$$(abc + 8l^3)\,\xi^2\,(by^3 - cz^3)\,[(abc + 2l^3)^2x^2 + 18bcl^4yz].$$

30. $$\overline{E} = -\tfrac{1}{9}\,\text{Jac}\,(\overline{K},\ U,\ \Lambda),\ = -\tfrac{1}{3}\begin{vmatrix} \partial_x\overline{K}, & X, & \xi \\ \partial_y\overline{K}, & Y, & \eta. \\ \partial_z\overline{K}, & Z, & \zeta \end{vmatrix}$$

and, if omitting in $\overline{K}$ the factor $abc + 8l^3$, we write $\overline{K} = Ax + By + Cz$, where

$$A = bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3 - 6al\eta^2\zeta^2, \text{ this is } = -\tfrac{1}{3}\begin{vmatrix} A, & X, & \xi \\ B, & Y, & \eta \\ C, & Z, & \zeta \end{vmatrix} \text{ which contains}$$

the term

$$\begin{aligned} \tfrac{1}{3}X(B\zeta - C\eta), &= \tfrac{1}{3}(ax^2 + 2lyz)\ \{\zeta\,(ca\eta^4 - 2ab\eta\zeta^3 - 2bc\eta\xi^3 - 6bl\zeta^2\xi^2) \\ &\qquad - \eta\,(ab\zeta^4 - 2bc\zeta\xi^3 - 2ca\zeta\eta^3 - 6cl\xi^2\eta^2)\}, \\ &= (ax^2 + 2lyz)(c\eta^3 - b\zeta^3)(2l\xi^2 + a\eta\zeta). \end{aligned}$$

Hence, restoring the factor $abc + 8l^3$, we have the terms

$$\overline{E} = (abc + 8l^3)\,\{x^2(c\eta^3 - b\zeta^3)\,[2al\xi^2 + a^2\eta\zeta] + yz\,(c\eta^3 - b\zeta^3)\,[4l^2\xi^2 + 2al\eta\zeta]\}.$$

31 and 32. $\qquad \overline{E}' = -\tfrac{1}{2}(\delta)\,\overline{E}, \qquad \overline{E}'' = -\tfrac{1}{8}(\delta^2)\,\overline{E}:$

$\overline{E}$ is of the form $(abc + 8l^3)(aU + bV + cW)$, and we operate with δ and δ^2 on the factors $2al\xi^2 + a^2\eta\zeta$, &c.; viz., $\delta(2al\xi^2 + a^2\eta\zeta) = 2(al' + a'l)\xi^2 + 2aa'\eta\zeta$, $\delta^2(2al\xi^2 + a^2\eta\zeta) = 4a'l'\xi^2 + 2a'^2\eta\zeta$, and we thus obtain in $\overline{E}'$ the term

$$(abc + 8l^3)\,x^2(c\eta^3 - b\zeta^3)\,[a(abc - 4l^3)\,\xi^2 - 6a^2l^2\eta\zeta],$$

and in $\overline{E}''$ the term

$$(abc + 8l^3)\,x^2(c\eta^3 - b\zeta^3)\,[-3al^2(abc + 2l^3)\,\xi^2 + 9a^2l^4\eta\zeta].$$

25. $M = \tfrac{1}{36}\,\text{Jac}\,(U, \Psi, \Lambda)$: this, omitting the factor $(abc + 8l^3)^2$ of Ψ is

$$= \tfrac{1}{2}\begin{vmatrix} ax^2 + 2lyz, & ax^2(ax^3 - 5by^3 - 5cz^3), & \xi \\ by^2 + 2lzx, & by^2(by^3 - 5cz^3 - 5ax^3), & \eta \\ cz^2 + 2lxy, & cz^2(cz^3 - 5ax^3 - 5by^3), & \zeta \end{vmatrix};$$

the coefficient of ξ herein is

$$\begin{aligned} &= \tfrac{1}{2}\,\{(bcy^2z^2 + 2clxz^3)(cz^3 - 5ax^3 - 5by^3) - (bcy^2z^2 + 2blxy^3)(by^3 - 5cz^3 - 5ax^3)\}, \\ &= \tfrac{1}{2}\,\{bcy^2z^2(-6by^3 + 6cz^3) + 2lx\,[-b^2y^6 + c^2z^6 + 5ax^3(by^3 - cz^3)]\}, \\ &= (by^3 - cz^3)[5alx^4 - blxy^3 - clxz^3 - 3bcy^2z^2]. \end{aligned}$$

Hence, restoring the factor $(abc + 8l^3)^2$, we have the term

$$(abc + 8l^3)^2\,.\,\xi\,(by^3 - cz^3)[5alx^4 - blxy^3 - clxz^3 - 3bcy^2z^2].$$

26. $\quad M' = -(\delta)\,M.$ Here M is of the form $(abc + 8l^3)^2(a^2U + \&c.)$; and the δ operates through the $(abc + 8l^3)^2a^2$, &c.; we in fact have in M' the term

$$-(abc + 8l^3)^2\,.\,\xi\,(by^3 - cz^3)[5al'x^4 - bl'xy^3 - cl'xz^3 - 3bc'y^2z^2],$$

which is

$$= (abc + 8l^3)^2\,.\,\xi\,(by^3 - cz^3)[(abc + 2l^3)(5ax^4 - bxy^3 - cxz^3) + 18bcl^2y^2z^2].$$

33. $$\overline{M} = -\tfrac{1}{6}[\text{Jac}](P,\ F,\ \Lambda), = -\tfrac{1}{6}\begin{vmatrix} -3lbc\xi^2 + (-abc + 4l^3)\eta\zeta, & \partial_\xi F, & x \\ -3lca\eta^2 + (-abc + 4l^3)\zeta\xi, & \partial_\eta F, & y \\ -3lab\zeta^2 + (-abc + 4l^3)\xi\eta, & \partial_\zeta F, & z \end{vmatrix},$$

and the whole coefficient of x is thus

$$= \tfrac{1}{6}\{[3lca\eta^2 + (abc - 4l^3)\zeta\xi]\,\partial_\zeta F - [3lab\zeta^2 + (abc - 4l^3)\xi\eta]\,\partial_\eta F\},$$

or substituting for $\tfrac{1}{6}\partial_\zeta F$, $\tfrac{1}{6}\partial_\eta F$ their values, this is

$$\begin{aligned} = \{3lca\eta^2 + (abc - 4l^3)\zeta\xi\}\,&[a^2b^2\zeta^5 - (abc + 16l^3)(b\zeta^2\xi^3 + a\zeta^2\eta^3) \\ &- 4l^2(bc\xi^4\eta + ca\xi\eta^4 + 4ab\xi\eta\zeta^3) - 8l(abc + 2l^3)\xi^2\eta^2\zeta] \\ - \{3lab\zeta^2 + (abc - 4l^3)\xi\eta\}\,&[a^2c^2\eta^5 - (abc + 16l^3)(a\eta^2\zeta^3 + c\eta^2\xi^3) \\ &- 4l^2(bc\zeta\xi^4 + 4ca\xi\eta^3\zeta + ab\xi\zeta^4) - 8l(abc + 2l^3)\xi^2\eta\zeta^2]. \end{aligned}$$

Collecting first the terms independent of $abc - 4l^3$, and next those which contain $abc - 4l^3$, each set contains the factor $c\eta^3 - b\zeta^3$, and the whole is $= c\eta^3 - b\zeta^3$ into

$$\begin{aligned} &-3la^2bc\eta^2\zeta^2 - 3a^2l(abc + 8l^3)\eta^2\zeta^2 - 12l^3(abc\xi^4 + a^2c\xi\eta^3 + a^2b\xi\zeta^3) - 24al^2(abc + 2l^3)\xi^2\eta\zeta \\ &\qquad + (abc - 4l^3)\{a^2c\xi\eta^3 + a^2b\xi\zeta^3 - (abc + 16l^3)\xi^4 + 12al^2\xi^2\eta\zeta\}; \end{aligned}$$

and here collecting the terms in ξ^4, $\xi(c\eta^3 + b\zeta^3)$, $\xi^2\eta\zeta$, and $\eta^2\zeta^2$, each of these contains the factor $abc + 8l^3$, and, finally, the term of $\overline{M}$ is

$$= (abc + 8l^3)(c\eta^3 - b\zeta^3)[(abc - 8l^3)\xi^4 - a^2c\xi\eta^3 - a^2b\xi\zeta^3 - 12al^2\xi^2\eta\zeta - 6a^2l\eta^2\zeta^2]\,x.$$

34. $$\overline{M'} = \tfrac{1}{6}(\delta)\,\overline{M}$$

$\overline{M}$ is here of the form $(abc + 8l^3)(aU + bV + cW)$; and, operating with δ through the $(abc + 8l^3)\,a$, &c., we obtain in $\overline{M'}$ the term

$$\tfrac{1}{6}(abc + 8l^3)\,x\,(c\eta^3 - b\zeta^3)[(a'bc + ab'c + abc' - 24l^2l')\,\xi^4 + \text{\&c.}],$$

where $a'bc + ab'c + abc' - 24l^2l' = 18abcl^2 + 24l^2(abc + 2l^3), = 6l^2(7abc + 8l^3)$, and the term thus is

$$= (abc + 8l^3)\,x\,(c\eta^3 - b\zeta^3)\,[(7abc + 8l^3)\,l^2\xi^4 \ldots].$$

This concludes the series of calculations.

CAMBRIDGE, ENGLAND, 17 *May*, 1881.

On Certain Expansion Theorems.

BY EMORY MCCLINTOCK, *F. I. A.*

I.

Any function fy can be expanded in terms (positive integral powers) of x, if $y = x\phi y$, provided fy and ϕy can be expanded in terms of y, and ϕy does not vanish with y; because x can, by dividing y by ϕy, be expressed in terms of y, and by reversion y, and therefore fy, in terms of x. These algebraic operations do not, however, disclose the law of the coefficients, which may be found in the following manner.

Assuming it known that $\mathrm{D}^{n-1}(\phi y)^n \mathrm{D} x^m_{[y=0]} = 0 \left(\text{where } \mathrm{D} = \frac{d}{dy}\right)$, unless $m = n$, when it becomes equal to $1 . 2 . 3 \ldots n$ or $n!$, we have, supposing f$y = a_0 + a_1 x + \frac{1}{1.2} a_2 x^2 + \ldots$,

$$\mathrm{D}^{n-1}(\phi y)^n \mathrm{D}\mathrm{f}y_{[y=0]} = \mathrm{D}^{n-1}(\phi y)^n \mathrm{D}(a_0 + a_1 x + \ldots) = \frac{1}{n!} a_n n! = a_n;$$

also f$0 = a_0$, so that

$$\mathrm{f}y = \mathrm{f}y_{[y=0]} + x . \phi y \mathrm{D}\mathrm{f}y_{[y=0]} + \frac{1}{2} x^2 . \mathrm{D}(\phi y)^2 \mathrm{D}\mathrm{f}y_{[y=0]} + \ldots . \tag{1}$$

For several reasons which will appear, I think this series will be found highly important. Lying midway between those of Lagrange and Bürmann, and transformable at once into either of those well-known series, the present result appears both simpler in form and easier of proof than either of them.

The correctness of the assumption upon which (1) is founded is readily shown. Since $x = y(\phi y)^{-1}$, or say $y\phi^{-1}$,

$$\mathrm{D}^{n-1}\phi^n \mathrm{D} x^m_{[y=0]} = \mathrm{D}^{n-1}\phi^n \mathrm{D} y^m \phi^{-m}_{[y=0]} = \mathrm{D}^{n-1}\phi^{n-m}\mathrm{D}y^m + \mathrm{D}^{n-1}\phi^n y^m \mathrm{D}\phi^{-m}_{[y=0]}.$$

If $m > n$, this vanishes, since y is a factor. If $m = n$, the second term vanishes, and there remains $\mathrm{D}^{n-1}\mathrm{D}y^n = n!$. If $m < n$, the whole expression again vanishes. We may write it $\mathrm{D}^{n-1}\phi^{n-m} m y^{m-1} - \mathrm{D}^{n-1} y^m \frac{m}{n-m} \mathrm{D}\phi^{n-m}_{[y=0]}$. If we suppose this expanded in terms of y, all of the terms necessarily vanish,

since $y=0$, except the term independent of y. Supposing c to be the coefficient of y^{n-m} in the expansion of ϕ^{n-m}, the independent term will be

$$\mathrm{D}^{n-1}cy^{n-m}my^{m-1} - \mathrm{D}^{n-1}y^{m}\frac{m}{n-m}(n-m)cy^{n-m-1} = mc\mathrm{D}^{n-1}y^{n-1} - mc\mathrm{D}^{n-1}y^{n-1} = 0.*$$

I have said that (1) can be transformed at once into Lagrange's theorem. If $u=y+z$, $\mathrm{D}_y fu = \mathrm{D}_z fu$. In (1), let $\mathrm{f}y = fu$, and $\phi y = \psi u$. Then $u = z + x\psi u$, and we have Lagrange's theorem, in its later form,

$$fu = fz + x\,.\,\psi z\mathrm{D}_z fz + \frac{1}{1\,.\,2}x^2\,.\,\mathrm{D}_z(\psi z)^2\mathrm{D}_z fz + \ldots.$$

If $x=1$, we have the same theorem in its original form,

$$fu = fz + \psi z\mathrm{D}_z fz + \ldots,$$

where $u = z + \psi u$. I have also said that (1) is simpler in form and easier of proof than Lagrange's theorem. As regards its form, it is almost identical with that case of Lagrange's theorem in which $z=0$, and, consequently, $u=y=x\phi y$,

$$\mathrm{f}y = \mathrm{f}z_{[z=0]} + x\,.\,\phi z\mathrm{D}_z\mathrm{f}z_{[z=0]} + \ldots,$$

a case declared by Lagrange himself, in comparison with that theorem, to be equally general and much more simple. Even this case of Lagrange's theorem is in form less simple than (1), because it introduces unnecessarily a third quantity, z, in addition to those given by the conditions of the problem, x and y. I am disposed to regard (1) as new and distinct, not only on account of this slight difference of form, but also, and chiefly, on account of its origin, diametrically opposite to that of the series just referred to; the one being a casual deduction from a more complex expression, retaining a remnant of its complexity in the unnecessary variable z, the other being, of and by itself, a simple and complete solution of the general problem of reversion. As regards demonstration, it is to be remarked that Lagrange's theorem has not been found easy to prove. By far the best and simplest demonstration, of many which I have met, is that of Laplace, usually followed by the text-books. Concerning it, we may observe that it assumes Maclaurin's theorem; that it proceeds, step by step, from one degree to the next higher; and that it keeps up a cross-fire of differentiations with respect to different variables. Concerning the proof of (1) we may remark that it involves, in the most elementary manner, a few of the most elementary

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* We might modify the demonstration by beginning with

$$\mathrm{D}^{n-1}\phi^n\mathrm{D}x^m = \mathrm{D}^n\phi^n x^m - \mathrm{D}^{n-1}x^m\mathrm{D}\phi^n,$$

both of which terms vanish when $m>n$, and the second when $m=n$, since $x=0$ when $y=0$, and x remains as a factor. When $m<n$, the term independent of y is

$$\mathrm{D}^{n}\ {}^{m}cy^{n-m} - \mathrm{D}^{n-1}y\ \frac{n}{n-m}\mathrm{D}cy^{n-m} = c\mathrm{D}^n y^n - c\mathrm{D}^{n-1}ny^{n-1} = 0.$$

principles, and announces at once the value of the general term. It has also the merit of indicating at the start the class of functions to which it is applicable. By means of it we may observe that Lagrange's theorem can be employed when fu and ψu can be expanded in terms of $y = u - z$, provided ψu does not vanish when $u = z$. Of course fu and ψu, or $f(y + z)$ and $\psi(y + z)$, can be expanded in terms of y in all cases to which Taylor's theorem applies.

I have said, on the other hand, that (1) can be transformed at once into Bürmann's theorem. Let x be any function of t, and let $\mathrm{f}y = ft$; then, since $\phi y = \frac{y}{x}$, (1) becomes Bürmann's first theorem,

$$ft = ft_{[y=0]} + x \,.\, \frac{y}{x}\mathrm{D}_y ft_{[y=0]} + \ldots .$$

I call this "first" because two other series sometimes get the name of "Bürmann's theorem." It was devised for the purpose of expanding one function of t in terms of another. Published in 1796, it was long regarded as not only different from, but more comprehensive than, that of Lagrange.*

Bürmann himself showed that Lagrange's theorem can be readily derived from his own. The converse, as will be seen, is equally true. Bürmann's proof of his theorem, the only demonstration of it with which I have met, may be found in Lacroix's Appendix and in Grunert's Supplement to Klügel's Wörterbuch, as well as in the works cited in the footnote. It occupies five of Grunert's pages. There can be no doubt of the correctness of my statement that (1) is both simpler in form and easier of proof than Bürmann's theorem.

We have, then, in (1) a series which is essentially identical with those of Lagrange and Bürmann, which is simpler in form than either, and which is at once a connecting link between them and a necessary step in the most direct demonstration of both. The inference seems warranted that we should regard it as the main proposition, of which the others are corollaries. This inference will be strengthened when we see how readily other corollaries can be deduced from it, which are usually referred to Lagrange's and Bürmann's theorems.

For instance, let $v = \psi(z + y)$, $\phi y = \chi v = \chi\psi(z + y)$, and $\mathrm{f}y = fv = f\psi(z + y)$. Then $v = \psi(z + x\chi v)$, and since, with regard to any function of $z + y$, $\mathrm{D}_y = \mathrm{D}_z$, we have from (1) Laplace's theorem,

$$fv = f\psi z + x \,.\, \chi\psi z \mathrm{D}_z f\psi z + \ldots .$$

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* See the Encyclopædia Metropolitana, Article *Integral Calculus*, part 3, p. 89; De Morgan, *Calculus*, pp. 303–305. In the appendix to the *Calculus*, p. 774, De Morgan says, however, "Bürmann's theorem is nothing but Lagrange's," and proves it concerning Bürmann's second theorem.

Again, let $y = u - a$, $\mathrm{f}y = fu$, $x = \dfrac{u-a}{\varphi(u-a)} = \psi u$, so that $\phi y = \dfrac{u-a}{\psi u}$, and we have from (1) what we may call Bürmann's second theorem,

$$fu = fu_{[u=a]} + \psi u \,.\, \frac{u-a}{\psi u}\, \mathrm{D} fu_{[u=a]} + \dots .$$

This "very remarkable formula," as Lacroix and others style it, gives explicitly the development of one function, fu, in terms of another, ψu.

Another of Bürmann's series may be derived from (1) by substituting for y, $u - a$; for x, $\psi u - \psi a$; for ϕy, $\dfrac{u-a}{\psi u - \psi a}$; and for $\mathrm{f}y$, fu.

Nearly seventy years ago, M. Wronski published a theorem for development, afterwards characterized by De Morgan as "excessively general," "elegant," "really remarkable." For this theorem, Professor Cayley has presented (*Quarterly Journal*, xii, 221) a simpler substitute, as follows:

"Suppose, in general, $\phi x = (x - a)\,\psi x$, or let the equation be $(x - a)\,\psi x + \lambda f x = 0$, that is, $x - a + \lambda \dfrac{fx}{\psi x} = 0$; we have then, by Lagrange's theorem,

$$Fx = F - \frac{\lambda}{1}\, F' \,\frac{f}{\psi} + \frac{\lambda^2}{1\,.\,2}\left\{ F'\left(\frac{f}{\psi}\right)^2 \right\}' - \frac{\lambda^3}{1\,.\,2\,.\,3}\left\{ F'\left(\frac{f}{\psi}\right)^3 \right\}'' + \text{\&c.}$$

Consider, for example, the term $\left\{ F'\left(\dfrac{f}{\psi}\right)^3 \right\}''$; this is $= \left\{ F'x \,.\, \dfrac{(x-a)^3 (fx)^3}{(\varphi x)^3} \right\}''$, the accents denoting differentiation in regard to x, and x being ultimately put $= a$; or, what is the same thing, it is $= \left(\dfrac{d}{d\theta}\right)^2 \left[F'(a + \theta)\, \dfrac{\theta^3 \{f(a+\theta)\}^3}{\{\varphi(a+\theta)\}^3} \right]$, the accents now denoting differentiation in regard to θ, and this being ultimately put $= 0$. This is

$$\left(\frac{d}{d\theta}\right)^2 \left[F'(a + \theta)\, \frac{\{f(a+\theta)\}^3}{(\varphi' a + \frac{\theta}{1\,.\,2}\,\varphi'' a + \dots)} \right].$$

This may be written $\left(F'f^3 \dfrac{1}{A^3}\right)''$, where $A = \phi' + \dfrac{1}{2}\,\theta\phi'' + \dfrac{1}{6}\,\theta^2\phi''' + \dots$, it being understood that as regards $F'f^3$, which is expressed as a function of a only (θ having been therein put $= 0$), the exterior accents denote differentiations in respect to a, whereas, in regard to A, $= \phi' + \dfrac{1}{2}\,\theta\phi'' + \text{\&c.}$, they denote differentiation in regard to θ, which is afterwards put $= 0$. And the theorem thus is

$$Fx = F - \frac{\lambda}{1}\left(F'f \,.\, \frac{1}{A}\right) + \frac{\lambda^2}{1\,.\,2}\left(F'f^2 \,.\, \frac{1}{A^2}\right)' - \frac{\lambda^3}{1\,.\,2\,.\,3}\left(F'f^3 \,.\, \frac{1}{A^3}\right)'' + \text{\&c.''}$$

The noteworthy result thus derived by Professor Cayley may be obtained from (1), in a very direct manner, as follows. Given $\psi z = xfz$, and $\psi a = 0$, to develop fz in terms of x. Let $y = z - a$, and

$$\phi y = \frac{y}{x} = \frac{yfz}{\psi z} = \frac{yf(y+a)}{\psi(y+a)} = \frac{f(y+a)}{\psi' a + \frac{1}{2}\, y\psi'' a + \ldots}.$$

Then (1) becomes

$$\mathrm{f}z = \mathrm{f}(y+a)_{[y=0]} + x\,.\,\phi y \mathrm{D}\mathrm{f}(y+a)_{[y=0]} + \frac{1}{1\,.\,2}\,x^2\,.\,\mathrm{D}(\phi y)^2 \mathrm{D}\mathrm{f}(y+a)_{[y=0]} + \ldots.$$

II.

Of the various corollaries following from (1), two are of great practical importance, and are apparently perfect in form and substance, incapable of amendment. I refer, of course, to Lagrange's and Laplace's theorems. The several series proposed by Bürmann, as aids in developing one function in terms of another, do not seem to be quite so perfect. They accomplish nothing which cannot be done, with suitable transformations, by either of those theorems or by (1). Their *raison d'être*, therefore, lies solely in the help which they afford by indicating what transformations may be necessary, and by formulating the results of such transformations. If we have to expand fu in terms of ψu, and if ψu happens to be readily divisible by $u - a$, Bürmann's second theorem will be immediately available and, of course, useful. If, as is much more likely, ψu is not conveniently divisible by $u - a$, we may perhaps turn to Bürmann's first theorem, which tells us to look for some function of u which vanishes with ψu, in a finite ratio. This is rather vague, and, as a matter of actual practice, most mathematicians would prefer to leave Bürmann's series alone, and to employ one of the other more familiar theorems, devising at the moment such transformations as might seem necessary. There is, however, a formula applicable to a large number of cases, which seems to have escaped the notice of Bürmann and other writers, and which would, I think, be found of use if borne in mind by those who have occasion to deal with developments of the kind in question.

Rule.—To expand fu in terms of fu, find some expression of the form $\psi^{-1}u - a$ which will exactly divide fu. Then

$$\mathrm{f}u = \mathrm{f}\psi a + fu\,.\,\frac{y-a}{f\psi y}\,\mathrm{D}\mathrm{f}\psi y + \frac{1}{1\,.\,2}\,(fu)^2\,.\,\mathrm{D}\left(\frac{y-a}{f\psi y}\right)^2 \mathrm{D}\mathrm{f}\psi y + \ldots_{[y=a]}, \qquad (2)$$

the general term being $\frac{1}{1\,.\,2\,.\,.\,.\,.\,m}(fu)^m.\,\mathrm{D}^{m-1}\left(\frac{y-a}{f\psi y}\right)^m \mathrm{D}\mathfrak{f}\psi y$. If, in applying this rule, we find that fu can be conveniently divided by $u-a$, the series becomes equivalent, as a special case, to Bürmann's second theorem, which it thus practically replaces. As an illustration, let $fu = u^6 - au^2$. Here fu is divisible by $u^4 - a$; so that $\psi^{-1}u = u^4$, $\psi u = u^{\frac{1}{4}}$, and $\frac{y-a}{f\psi y} = (y-a) \div (y^{\frac{3}{2}} - ay^{\frac{1}{2}}) = y^{-\frac{1}{2}}$. Hence,

$$\mathfrak{f}u = \mathfrak{f}a^{\frac{1}{4}} + fu\,.\,a^{-\frac{1}{2}}\,\mathrm{D}_a\mathfrak{f}a^{\frac{1}{4}} + \frac{1}{1\,.\,2}(fu)^2.\,\mathrm{D}_a a^{-1}\mathrm{D}_a\mathfrak{f}a^{\frac{1}{4}} + \ldots .$$

The series (2) is derived from (1) by putting $x = fu$, $y = \psi^{-1}u - a$, $\left(\text{whence } u = \psi(y+a),\ \phi y = \frac{y}{x} = \frac{y}{f\psi(y+a)},\ \mathfrak{f}y = \mathfrak{f}u = \mathfrak{f}\psi(y+a)\right)$, and finally writing $y - a$ for y, throughout. It may also be derived very readily from Laplace's theorem, to which it bears exactly the same relation as that borne to Lagrange's by Bürmann's second theorem. Whether there is any real necessity for an explicit formula, distinct from Laplace's theorem, for dealing with this subject, may be questioned. Bürmann and many other writers have thought it desirable to have one or more. The present formula appears to meet the requirement.

III.

Among the questions to which (1) is directly applicable, one of the most important is that of the common reversion of series. Expressed most simply, this question is, given $x = y + ay^2 + by^3 + \ldots$, to expand y in terms of x. Here $y = x\phi y$, where ϕy is the reciprocal of $1 + ay + by^2 + \ldots$, or $(1 + ay + \ldots)^{-1}$. and from (1) we have

$$y = x + \frac{1}{2}x^2.\,\mathrm{D}\,(\phi y)^2_{[y=0]} + \ldots .$$

This is the same result, of course, as that usually derived from Lagrange's theorem, with the merely formal difference that y takes the place of z in the second member. The determination of the coefficients is a matter of some difficulty; so much so that on several occasions on which extensions have been made to the known portion of the series, the name of the calculator has been recorded, The latest calculation appears to have been that made by De Morgan (Penny Cyclopædia, Article *Reversion of Series*), who describes the process devised by him to facilitate the work.

It has, perhaps, not been noticed that, having given the series

$$\frac{1}{x} = \frac{1}{y} + \alpha y + \beta y^2 + \ldots,$$

it is peculiarly easy to expand y in terms of x. Here we have, as before, $y = x\phi y$, and

$$y = x + \frac{1}{2} x^3 . \text{D} (\phi y)^3_{[y=0]} + \ldots,$$

but in this case $\phi y = 1 + \alpha y^3 + \beta y^3 + \ldots$, and the coefficients are simpler and easier to determine. In the former instance,

$y = x - ax^2 - (b - 2a^2)x^3 - (c - 5ab + 5a^3)x^4 - (d - 3b^2 - 6ac + 21a^2b - 14a^4)x^5 - \ldots;$

while in the present,

$$y = x + \alpha x^3 + \beta x^4 + (\gamma + 2\alpha^2)\, x^5 + (\delta + 5\alpha\beta)\, x^6 + \ldots. \tag{3}$$

Either result may be derived from the other by putting $a = -\alpha x$, $b = -\beta x$, &c. In another light, we may regard (3) as the development of a root of the equation $1 - \frac{y}{x} + \alpha y^3 + \beta y^3 + \ldots = 0$, in descending powers of the coefficient of y. When this equation has a very small root, the series may be employed for its arithmetical computation, though it is not so good for that purpose as the other series, produced by reversion.

My chief purpose in presenting (3) is to show how it may be employed in facilitating the determination of the usual reversion-coefficients, by a method even less laborious than that pursued by De Morgan. The reason why the coefficients of (3) are, in comparison with those of the other series, so easy to determine is two-fold: first, ϕy is not a fraction, and secondly, $\text{D}\phi y$ vanishes when $y = 0$. In consequence of these favoring circumstances, the coefficients may, by the help of the combinatorial analysis (see De Morgan, *Calculus*, pp. 335, 336), be written down almost at will. To find the coefficient of x^m, let us first see how y^{m-1} can be resolved into factors of the form $y^2, y^3, \ldots y^{m-1}$. If for example, $m = 10$, we find that y^9 can be resolved in several ways, as follows: y^9, y^2y^7, y^3y^6, y^4y^5, $y^2y^2y^5$, $y^2y^3y^4$, $y^3y^3y^3$, $y^2y^2y^2y^3$. Each of these expressions stands for a term or element of the result, it being understood that instead of each power of y we must write its coefficient in $1 - \frac{y}{x} + \alpha y^2 + \beta y^3 + \ldots = 0$. Thus θ, $\alpha\zeta$, $\beta\varepsilon$, $\gamma\delta$, $\alpha^2\delta$, $\alpha\beta\gamma$, β^3, and $\alpha^3\beta$ will be elements in the case chosen. To each element thus found, containing more than one factor, a numerical coefficient is to be annexed. Each element being of the form $\alpha^r\beta^s\gamma^t \ldots$, its coefficient will be

$\frac{(m-1)(m-2)\ldots(m-n+1)}{r!\,s!\,t!\ldots}$, where $n = r + s + t + \ldots$. Thus, the coefficient of $\alpha\beta\gamma$ is $9 \cdot 8 = 72$; that of $\alpha^2\delta$ is $\frac{9 \cdot 8}{2} = 36$, and so in other cases. The result, in the case chosen, may thus be written out:

Coefficient of $x^{10} = \theta + 9(\alpha\zeta + \beta\varepsilon + \gamma\delta) + 36\alpha^2\delta + 72\alpha\beta\gamma + 12\beta^3 + 84\alpha^3\beta$.

If, therefore, it is desired to determine the reversion-coefficients as far as those of x^m inclusive, we must find the coefficients of (3) in this way as far as the mth power, and also certain elements of the coefficients of higher powers (namely, those elements of the coefficient of each higher power, say x^{m+r}, for which n is not less than r), and then put $\alpha = -\frac{a}{x}$, $\beta = -\frac{b}{x}$, and so on.

The process thus indicated will give the expansion of y, where $x = y + ay^2 + by^3 + \ldots$. If the given series is $x = c_1 y + c_2 y^2 + c_3 y^3 + \ldots$, it can be reduced to the first form by dividing both members by c_1, when y can be expanded in terms of $\frac{x}{c_1}$.

IV.

Lagrange's and Laplace's theorems can be demonstrated by the means employed in proving (1), without explicitly naming the latter as a step in the process. In doing so, however, we cannot fail to be impressed with the fact that we are not treating the subject in the best and simplest manner. For Laplace's theorem, the proof would be as follows.

If $f\psi(y+z)$ and $\phi\psi(y+z)$ can be expanded in terms (positive integral powers) of y, and if $\phi\psi z$ is not 0, and if $x = y \div \phi\psi(y+z)$, then can x be expanded in terms of y, and by reversion y, and therefore $f\psi(y+z)$, in terms of x. It is required to determine the coefficients in the latter case. That is to say, it is required to determine the general form of a_n in

$$f\psi(y+z) = a_0 + x \cdot a_1 + \frac{1}{1 \cdot 2} x \cdot a_2 + \ldots.$$

Let it be shown, as before, that the expression $\mathrm{D}_y^{n-1}[\phi\psi(y+z)]^n \mathrm{D}_y x^m$ vanishes when $y = 0$, unless $m = n$, when it is equal to $n!$. Then

$$\mathrm{D}_y^{n-1}[\phi\psi(y+z)]^n \mathrm{D} f\psi(y+z)_{[y=0]} = \mathrm{D}_y^{n-1}[\phi\psi(y+z)]^n \mathrm{D}(a_0 + x \cdot a_1 + \ldots)$$

$$= \frac{1}{n!} a_n n! = a_n.$$

Since $\mathrm{D}_y f(y + z) = \mathrm{D}_z f(y + z)$, this expression is equivalent to

$$a_n = \mathrm{D}_z^{n-1}(\phi\psi z)^n \mathrm{D f}\psi z.$$

Also, $a_0 = \mathrm{f}\psi z$, since $y = 0$ when $x = 0$. Let $u = \psi(y + z)$; then, since $y = x\phi\psi(y + z) = x\phi u$, we have $u = \psi(z + x\phi u)$, and our first equation becomes

$$\mathrm{f}u = \mathrm{f}\psi z + x \,.\, \phi\psi z \mathrm{D}_z \mathrm{f}\psi z + \ldots,$$

which is Laplace's theorem.

For Lagrange's theorem, the proof would be the same, omitting throughout the functional symbol ψ.

MILWAUKEE, *May* 16, 1881.

Some Theorems in Numbers.

BY O. H. MITCHELL,

Fellow in the Johns Hopkins University.

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§1. *On the Residues mod. k of the Symmetric Functions of the Numbers less than k, where $k =$ any Integer.*

In the following $\equiv$ will denote identical congruity, *i. e.*, $\phi(x) \equiv \psi(x)$, mod. k, will denote that the coefficients of corresponding powers of x are respectively congruous to one another mod. k. If $\Pi(x-\alpha)$ denote the product $(x-1)(x-2)(x-3)\ldots.(x-a+1)$, where a is a prime number, we know that $\Pi(x-\alpha) \equiv x^{a-1}-1$ mod. a. This expresses the fact that the symmetric functions $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, &c. of the numbers less than (and prime to) a are each $\equiv 0$ mod. a, except the last, which is $\equiv -1$ mod. a. It is proposed in this section to determine the value of $\Pi(x-\alpha)$ for any integer, k, $= a^t b^u \ldots g^v h^w \ldots q^z$, where a, b, &c. are prime numbers; or, more generally, to determine the residue, mod. k, of $\Pi(x-\theta_s)$, where θ'_s, θ''_s, θ'''_s, &c. are those numbers less than k which contain $s = hi\ldots q$, and no prime factor of k not found in s. These numbers were called in my former paper, Vol. 3, No. 4, of this Journal, the s-totitives of k. The number of them was called the s-totient of k, and denoted by $\tau_s(k)$. It was there seen that $\tau_s(k) = a^{t-1}b^{u-1}\ldots q^{z-1}(a-1)(b-1)\ldots(g-1)$.

Theorem I. If $\Pi_{a^t}(x-\theta_1)$ denote the product $(x-\theta'_1)(x-\theta''_1)(x-\theta'''_1)\ldots$, where the numbers θ_1 are the prime totitives of a^t, a being an odd prime number, then

$$\Pi_{a^t}(x-\theta_1) \equiv (x^{a-1}-1)^{a^{t-1}} \text{ mod. } a^t;$$

and if θ'_a, θ''_a, &c. be the a-totitives of a^t, then

$$\Pi_{a^t}(x-\theta_a) \equiv x^{a^{t-1}} \text{ mod. } a^t.$$

To prove the theorem it will be shown first that

$$\Pi_{a^t}(x-\theta_1) \equiv [\Pi_{a^{t-1}}(x-\theta_1)]^a \text{ mod. } a^t.$$

Let $\alpha, \beta, \gamma, \ldots \omega$ be the prime totitives of a^{t-1}; then the prime totitives of a^t will be given by $\alpha + \lambda a^{t-1}$, $\beta + \lambda a^{t-1}$, &c. where λ has all values from 0 to $a-1$. Now

$$\Pi\,(x - \overline{\alpha + \lambda a^{t-1}}) \equiv x^a - (C_1^a \alpha + A a^{t-1})\,x^{a-1} + (C_2^a \alpha^2 + C_1^{a-1} A a^{t-1} \alpha)\,x^{a-2}$$
$$- (C_3^a \alpha^3 + C_2^{a-1} A a^{t-1} \alpha^2)\,x^{a-3} + \&\text{c. mod. } a^t,$$

where the terms containing $a^{2(t-1)}$, $a^{3(t-1)}$, &c. have been dropped, and C_1^a, C_2^a, &c. are the successive binomial coefficients of the a^{th} power, and $A = 0 + 1 + 2 + 3 + \ldots + (a-1) = \dfrac{a(a-1)}{2}$. But $A \equiv 0$ mod. a, $\therefore A a^{t-1} \equiv 0$ mod. a^t, and

$$\Pi\,(x - \overline{\alpha + \lambda a^{t-1}}) \equiv (x - \alpha)^a \text{ mod. } a^t,$$
$$\therefore \Pi_{a^t}(x - \theta_1) \equiv (x-\alpha)^a (x-\beta)^a \ldots (x-\omega)^a \equiv [\Pi_{a^{t-1}}(x - \theta_1)]^a \text{ mod. } a^t.$$

Now,

$$\Pi_a(x - \theta_1) \equiv x^{a-1} - 1 \quad \text{mod. } a,$$
$$\therefore \Pi_{a^2}(x - \theta_1) \equiv (x^{a-1} - 1)^a \quad \text{“} \quad a^2,$$
$$\therefore \Pi_{a^3}(x - \theta_1) \equiv (x^{a-1} - 1)^{a^2} \quad \text{“} \quad a^3,$$
$$\therefore \Pi_{a^t}(x - \theta_1) \equiv (x^{a-1} - 1)^{a^{t-1}} \quad \text{“} \quad a^t,$$

and the first part of the theorem is proved. The same proof applies to the second part of the theorem, if we suppose α, β, &c. to be not the prime totitives, but the a-totitives of a^{t-1}. Then we get, just as before,

$$\Pi_{a^t}(x - \theta_a) \equiv [\Pi_{a^{t-1}}(x - \theta_a)]^a \text{ mod. } a^t.$$

But we have

$$\Pi_a\,(x - \theta_a) = x - 0\,,$$
$$\therefore \Pi_{a^2}(x - \theta_a) \equiv x^a \text{ mod. } a^2, \ \&\text{c.},$$
$$\therefore \Pi_{a^t}(x - \theta_a) \equiv x^{a^{t-1}} \text{ mod. } a^t. \quad \text{Q. E. D.}$$

If, in the foregoing, we put $a = 2$, then

$$\Pi\,(x - \overline{\alpha + 2^{t-1}\lambda}) \equiv x^2 - (2\alpha \pm 2^{t-1})\,x + (\alpha^2 \pm 2^{t-1}) \text{ mod. } 2^t,$$

since $A = 0 + 1 \equiv \pm 1$ mod. 2. This may be written

$$\Pi\,(x - \overline{\alpha + 2^{t-1}\lambda}) \equiv (x-\alpha)^2 \pm 2^{t-1}(x-1) \text{ mod. } 2^t,$$
$$\therefore \Pi_{2^t}(x - \theta_1) \equiv \{(x-\alpha)^2 \pm 2^{t-1}(x-1)\}\{(x-\beta)^2 \pm 2^{t-1}(x-1)\} \ldots$$
$$\{(x-\omega)^2 \pm 2^{t-1}(x-1)\} \text{ mod. } 2^t,$$
$$\therefore \Pi_{2^t}(x - \theta_1) \equiv [\Pi_{2^{t-1}}(x-\alpha)]^2 \pm 2^{t-1}(x-1)\{(x-\beta)^2(x-\gamma)^2 \ldots (x-\omega)^2$$
$$+ (x-\alpha)^2(x-\gamma)^2 \ldots (x-\omega)^2 + \ldots + (x-\alpha)^2(x-\beta)^2(x-\gamma)^2 \ldots (x-\psi)^2\} \text{ mod. } 2^t.$$

Now, if $t > 2$, the number of the prime totitives of 2^{t-1} is even, *i. e.*, α, β, &c. are the 2^{t-2} odd numbers from 1 to $2^{t-1} - 1$. Thus, we have

$$\Pi_{2^t}(x - \theta_1) \equiv [\Pi_{2^{t-1}}(x - \theta_1)]^2 \pm 2^{t-1}(x-1)\{2^{t-2}(x-1)^{2^{t-1}-2} + 2f(x)\} \text{ mod. } 2^t$$
$$\therefore \Pi_{2^t}(x - \theta_1) \equiv [\Pi_{2^{t-1}}(x - \theta_1)]^2 \text{ mod. } 2^t, \text{ when } t > 2\,.$$

If α, β, γ, &c. be the 2-totitives of 2^{t-1}, *i. e.*, the 2^{t-2} even numbers from 0 to $2^{t-1}-2$, then we get, as before,

$$\Pi_{2^t}(x-\theta_2) \equiv [\Pi_{2^{t-1}}(x-\theta_2)]^2 \pm 2^{t-1}(x-1)\{2^{t-2}x^{2^{t-1}-2} + 2\phi(x)\} \text{ mod. } 2^t$$

$$\therefore\ \Pi_{2^t}(x-\theta_2) \equiv [\Pi_{2^{t-1}}(x-\theta_2)]^2 \text{ mod. } 2^t, \text{ when } t > 2.$$

By inspection, we have

$$\Pi_2(x-\theta_1) = x-1,$$
$$\Pi_4(x-\theta_1) \equiv x^2-1 \text{ mod. } 4,$$
$$\therefore\ \Pi_8(x-\theta_1) \equiv (x^2-1)^2 \text{ mod. } 8,$$

and thence, by induction,

$$\Pi_{2^t}(x-\theta_1) \equiv (x^2-1)^{2^{t-2}} \text{ mod. } 2^t.$$

So we have, by inspection,

$$\Pi_2(x-\theta_2) = x,$$
$$\Pi_4(x-\theta_2) = x(x-2)$$
$$\therefore\ \Pi_8(x-\theta_2) \equiv x^2(x-2)^2 \text{ mod. } 8,$$
$$\therefore\ \Pi_{16}(x-\theta_2) \equiv x^4(x-2)^4 \quad \text{''} \quad 16,$$
$$\therefore\ \Pi_{2^t}(x-\theta_2) \equiv [x(x-2)]^{2^{t-2}} \text{ mod. } 2^t.$$

This last expression may evidently be written

$$\Pi_{2^t}(x-\theta_2) \equiv x^{2^{t-1}-2}(x^2 + 2^{t-1}x + 2^{t-1}) \text{ mod. } 2^t.$$

Thus we have

Theorem II. If $\Pi(x-\theta_1)$ denote the continued product $(x-\theta_1')(x-\theta_1'')\ldots$, where the numbers θ_1 are the successive odd numbers from 1 to 2^t-1, then, t being > 1,

$$\Pi(x-\theta_1) \equiv (x^2-1)^{2^{t-2}} \text{ mod. } 2^t;$$

and if $\Pi(x-\theta_2)$ denote a similar product where the numbers θ_2 are the successive even numbers from 0 to 2^t-2, then, t being > 2,

$$\Pi(x-\theta_2) \equiv [x(x-2)]^{2^{t-2}} \equiv x^{2^{t-1}-2}(x^2 + 2^{t-1}x + 2^{t-1}) \text{ mod. } 2^t.$$

Example of Theorem I. Suppose $a^t = 27$. Then

$$(x-1)(x-2)(x-4)(x-5)(x-7)(x-8)(x-10)(x-11)(x-13)$$
$$(x+13)(x+11)\ldots(x+1)$$
$$\equiv (x^2-1)(x^2-10)(x^2-19)(x^2-4)(x^2-7)(x^2-16)(x^2-22)(x^2-25)(x^2-13)$$
$$\equiv (x^2-1)^3(x^4-11x^2+1)^3 \equiv (x^6-12x^4+12x^2-1)^3 \equiv (x^2-1)^9 \text{ mod. } 27.$$

That is, $\Pi(x-\theta_1) \equiv (x^2-1)^{3^2}$ mod. 3^3. So,

$$\Pi(x-\theta_3) = (x-0)(x-3)(x-6)\ldots(x+6)(x+3) \equiv x^9 \text{ mod. } 27.$$

Example of Theorem II. Suppose $2^t = 16$. Then

$$\Pi(x-\theta_1) = (x-1)(x-3)(x-5)(x-7)(x-9)(x-11)(x-13)(x-15)$$
$$\equiv (x^2-1)^2(x^2-9)^2 \equiv (x^4-10x^2+9)^2 \equiv x^8-4x^6+6x^4-4x^2+1 = (x^2-1)^4 \text{ mod. } 16.$$

So, $(x-0)(x-2)(x-4)(x-6)(x-8)(x+6)(x+4)(x+2) \equiv x^3(x-8)(x^2-4)^2$
$\equiv x^6(x^2+8x+8) \equiv [x(x-2)]^4$ mod. 16.

In the following, if $k = a^t b^u \,.\,.\, g^v h^w \,.\,.\, q^z$, and $s = hi \,.\,.\, q$, R_s will denote that one of the roots of $x^2 \equiv x$ mod. k, which is an s-totitive of k. (For the properties of these roots or repetents, see my paper, Vol. III, No. 4, of this Journal). $R_{\bar{s}}$ will denote that repetent of k whose subscript contains all those prime factors of k not found in s. Thus $R_{\bar{q}}$ will denote briefly the same thing as $R_{ab\ldots p}$. We can now prove the general theorem concerning the function $\Pi_k(x-\theta_s)$ of any integer k.

Theorem III. If $k = a^t b^u \,.\,.\, g^v h^w \,.\,.\, q^z$, where a, b, &c. are different prime umbers, and if $s = h \ldots q$, $\sigma = h^w \ldots q^z$, and θ'_s, θ''_s, &c. be the s-totitives of k, then

$$\Pi_k(x-\theta_s) \equiv R_{\bar{a}}(x^{a-1}-1)^{\tau_{as}(k)} + R_{\bar{b}}(x^{b-1}-1)^{\tau_{bs}(k)} + \ldots + R_{\bar{g}}(x^{g-1}-1)^{\tau_{gs}(k)}$$
$$+ R_{\bar{s}} x^{\tau_s(k)} \text{ mod. } k;$$

except (1) when $\dfrac{k}{\sigma} = 4$ or a higher power of 2, in which case

$$\Pi_k(x-\theta_s) \equiv R_{\bar{2}}(x^2-1)^{\frac{1}{2}\tau_{2s}(k)} + R_{\bar{s}} x^{\tau_s(k)} \text{ mod. } k,$$

and (2) when $\dfrac{k}{\sigma} = 1$ and $a^t = 2^t$ where $t =$ or > 2, in which case, if $t = 2$, then

$$\Pi_k(x-\theta_s) \equiv x^{\tau_s(k)} + \tfrac{1}{2} k x^{\tau_s(k)-1} \text{ mod. } k,$$

and if $t > 2$, then

$$\Pi_k(x-\theta_s) \equiv x^{\tau_s(k)} + \tfrac{1}{2} k x^{\tau_s(k)-1} + \tfrac{1}{2} k x^{\tau_s(k)-2} \text{ mod. } k.$$

To prove the theorem, let us consider each component of the modulus separately. With respect to mod. a^t, the formula to be proved reduces to

$$\Pi_k(x-\theta_s) \equiv (x^{a-1}-1)^{\tau_{as}(k)} \text{ mod. } a^t,$$

since $R_{\bar{b}}$, $R_{\bar{c}}$, $\ldots R_{\bar{g}}$, $R_{\bar{s}}$ each contain a^t, and $R_{\bar{a}} \equiv 1$, mod. a^t. Now it is clear that if we take the residues mod. a^t of the numbers θ_s we shall get each prime totitive of a^t the same number of times, and shall consequently get, in all, $\dfrac{\tau_s(k)}{\tau_1(a^t)}$ groups, each group containing all the prime totitives of a^t. If

$\Pi_a\,(x-\theta_s)$ denote the product taken for those numbers θ_s which compose any one of these groups, then by Theorem I

$$\Pi_{a^t}(x-\theta_s)\equiv(x^{a-1}-1)^{a^{t-1}} \text{ mod. } a^t,$$

if a be an odd prime number. Hence

$$\Pi_k(x-\theta_s)\equiv(x^{a-1}-1)^{a^{t-1}\frac{\tau_s(k)}{\tau(a^t)}} \text{ mod } a^t,$$

or

$$\Pi_k\,(x-\theta_s)\equiv(x^{a-1}-1)^{\tau_{as}(k)} \text{ mod. } a^t.$$

If $a=2$, we have, by Theorem II,

$$\Pi_{2^t}(x-\theta_s)\equiv(x^2-1)^{2^{t-2}} \text{ mod. } 2^t,$$

$$\therefore\ \Pi_k\,(x-\theta_s)\equiv(x^2-1)^{2^{t-2}\frac{\tau_s(k)}{2^{t-1}}} \text{ mod. } 2^t.$$

Now, if $\frac{k}{\sigma}$ contain at least one odd prime number, then $2^{t-2}.\frac{\tau_s(k)}{2^{t-1}}\left(=\frac{1}{2}\,\tau_{2s}(k)\right)$ is divisible by 2^{t-1}; but $(x^2-1)^{2^{t-1}}=(x-1)^{2^{t-1}}(\overline{x-1}+2)^{2^{t-1}}$, and $(\overline{x-1}+2)^{2^{t-1}}\equiv(x-1)^{2^{t-1}}$ mod. 2^t, therefore $(x^2-1)^{2^{t-1}}\equiv(x-1)^{2^t}$ mod. 2^t. Therefore,

$$\Pi_k\,(x-\theta_s)\equiv(x-1)^{\tau_{2s}(k)} \text{ mod. } 2^t.$$

But if $\frac{k}{\sigma}=4$, or a higher power of 2, according to exception (1), then we have, as above,

$$\Pi_k\,(x-\theta_s)\equiv(x^2-1)^{\frac{1}{2}\tau_{2s}(k)} \text{ mod. } 2^t.$$

Having shown that the formula holds true for mod. a^t, we have shown it true for mod. $\frac{k}{\sigma}$, since no distinction is to be made among $a,\ b,\ \ldots\ g$.

Let us now consider one of the components of σ, as q^s. The formula to be proved reduces to

$$\Pi_k\,(x-\theta_s)\equiv x^{\tau_s(k)} \text{ mod. } q^s,$$

since $R_{\bar{s}}\equiv 1$ mod. q^s, and $R_{\bar{a}}\ R_{\bar{b}},\ \ldots\ R_{\bar{g}}$ each contain q^s. The numbers θ_s all contain q, and it is clear that, if we take their residues mod. q^s, we shall get each q-totitive of q^s the same number of times, and shall consequently get, in all, $\frac{\tau_s(k)}{q^{s-1}}$ groups, each group containing all the q-totitives of q^s. If $\Pi_{q^s}(x-\theta_s)$ denote the product taken for those numbers θ_s which compose any one of these groups, then, by Theorem I, we have, when q is odd,

$$\Pi_{q^s}(x-\theta_s)\equiv x^{q^{s-1}} \text{ mod. } q^s.$$

$$\therefore\quad \Pi_k\ (x-\theta_s)\equiv x^{\tau_s(k)} \text{ mod. } q^s.$$

But if $q=2$, we have, by Theorem II, second part,

$$\Pi_{2^s}(x-\theta_s)\equiv(x(x-2))^{2^{s-2}} \text{ mod. } 2^s.$$

$$\therefore\ \Pi_k\,(x-\theta_s)\equiv(x(x-2))^{\frac{1}{2}\tau_s(k)} \text{ mod. } 2^s.$$

And if $\frac{k}{\sigma}$ be not equal to unity, then $\frac{1}{2}\tau_s(k)$ will be divisible by 2^{s-1} when $q = 2$. Then, since $[x(x-2)]^{2^{s-1}} \equiv x^{2^s} \bmod. 2^s$, it follows that

$$\Pi_k(x-\theta_s) \equiv (x)^{\tau_s(k)} \bmod. 2^s, \text{ as before.}$$

But if $\frac{k}{\sigma} = 1$ and $q = 2$, then $\frac{\tau_s(k)}{2^{s-1}}$ is odd, and we have, by Theorem II, as before,

$$\Pi_{q^s}(x-\theta_s) \equiv [x(x-2)]^{2^{s-2}} \equiv x^{2^{s-1}-2}(x^2 + 2^{s-1}x + 2^{s-1}) \bmod 2^s,$$

where the last term in the parenthesis is present or absent according as $z > 2$ or $z = 2$.

$$\therefore \Pi_k(x-\theta_s) \equiv [x^{2^{s-1}-2}(x^2 + 2^{s-1}x + 2^{s-1})]^{\frac{\tau_s(k)}{2^{s-1}}} \bmod. 2^s.$$

Now, $(x^2 + 2^{s-1}x + 2^{s-1})^n \equiv (x^2)^n + n\{(x^2)^{n-1}(2^{s-1}x) + (x^2)^{n-1}(2^{s-1})\} \bmod. 2^s$ which, when n is odd, becomes

$$\equiv x^{2n} + 2^{s-1}x^{2n-1} + 2^{s-1}x^{2n-2} = x^{2n-2}(x^2 + 2^{s-1}x + 2^{s-1}) \bmod. 2^s.$$

Then, since $\frac{\tau_s(k)}{2^{s-1}}$ is odd, we have for $z > 2$,

$$\Pi_k(x-\theta_s) \equiv x^{n(2^{s-1}-2)}(x^2 + 2^{s-1}x + 2^{s-1})\,x^{2n-2} \bmod. 2^s,$$

where $n = \frac{\tau_s(k)}{2^{s-1}}$. Now, since $\frac{k}{2^s}$ is odd, we have $\frac{1}{2}k = 2^{s-1}(2\lambda + 1)$. Then $2^{s-1} \equiv \frac{1}{2}k \bmod. 2^s$.

$$\therefore \Pi_k(x-\theta_s) \equiv x^{\tau_s(k)-2}\left(x^2 + \tfrac{1}{2}kx + \tfrac{1}{2}k\right) \bmod. 2^s,$$

which is exception (2), second part. In exactly the same way we get when $z = 2$,

$$\Pi_k(x-\theta_s) \equiv x^{\tau_s(k)-2}\left(x^2 + \tfrac{1}{2}kx\right) \bmod. 2^s,$$

which is exception (2), first part.

Having shown, now, that the formula of the theorem holds good with respect to any component of the modulus, the theorem is proved.

In the expansion of $R_{\bar a}(x^{a-1} - 1)^{\tau_a(k)}$ the coefficient of x^r is

$$\equiv (-)^{\frac{\tau_a(k)-r}{a-1}} R_{\bar a}\, C_{\frac{r}{a-1}}^{\frac{\tau_a(k)}{a-1}} \text{ or } = 0,$$

according as r is or is not divisible by $a - 1$, where $C_{\frac{r}{a-1}}^{\frac{\tau_a(k)}{a-1}}$ is a binomial coeffi-

cient. For, if r contain $a-1$, then $(-)^{\tau_a(k)-\frac{r}{a-1}} = (-)^{\frac{\tau_s(k)-r}{a-1}}$. Therefore, putting $\tau_s(k) - r = m$, the preceding theorem may be stated as follows:

Theorem IV. If $k = a^t b^u \ldots g^v h^w \ldots q^z$, $s = h \ldots q$, and $P_m(\theta_s)$ denote $\Sigma \theta'_s \theta''_s \ldots \theta_s^{[m]}$, where θ'_s, &c. are the s-totitives of k, then

$$P_m(\theta_s) \equiv \sum_{\omega=a}^{\omega=g} (-)^{\frac{m}{\omega-1}} R_{\bar{\omega}} C^{\tau_{\omega s}(k)}_{\frac{m}{\omega-1}} \text{ mod. } k,$$

the summation including only those terms for which $\frac{m}{\omega-1}$ is an integer; except (1) when $\frac{k}{\sigma} = 4$ or a higher power of 2, in which case

$$P_m(\theta_s) \equiv (-)^{\frac{m}{2}} R_{\bar{2}} C^{\frac{1}{2}\tau_{2s}(k)}_{\frac{m}{2}} \text{ mod. } k,$$

and (2) when $\frac{k}{\sigma} = 1$ and one of the components of $k = 2^n$, in which case, if $n = 2$, then

$$P_1(\theta_s) \equiv \tfrac{1}{2} k, \text{ and } P_{1+\lambda} \equiv 0 \text{ mod. } k,$$

but if $n > 2$, then

$$P_1(\theta_s) \equiv P_2(\theta_s) \equiv \tfrac{1}{2} k, \text{ and } P_{2+\lambda} \equiv 0 \text{ mod. } k.$$

When $m = \tau_s(k)$, the formula becomes

$$P_{\tau_s(k)}(\theta_s) \equiv R_{\bar{a}} + R_{\bar{b}} + \ldots + R_{\bar{g}} \equiv R_{\overline{ab\ldots g}} = R_s \text{ mod. } k,$$

except when $\frac{k}{\sigma} = p^n$, $2p^n$, or 4, and $\frac{\sigma}{s}$ is at the same time an odd number, in which case $\tau_s(k) : p-1$ is odd, and the formula becomes

$$P_{\tau_s(k)}(\theta_s) \equiv -R_s \text{ mod. } k.$$

This special case of the formula is the generalization of the Wilsonian theorem given in my former paper, for $P_{\tau_s(k)}(\theta_s) =$ the product of the s-totitives of k.

Example. Suppose $k = 60 = 2^2.3.5$, then those three roots of $x^2 \equiv x$ mod. 60 which I have denoted by $R_{\bar{2}}$, $R_{\bar{3}}$, $R_{\bar{5}}$ are 45, 40, 36 respectively.

I. $s = 1$.

$$P_m(\theta_1) \equiv (-)^{\frac{m}{2-1}} R_{\bar{2}} C^{\tau_2(60)}_{\frac{m}{2-1}} + (-)^{\frac{m}{3-1}} R_{\bar{3}} C^{\tau_3(60)}_{\frac{m}{3-1}} + (-)^{\frac{m}{5-1}} R_{\bar{5}} C^{\tau_5(60)}_{\frac{m}{5-1}} \text{ mod. } 60.$$

Whence

$$P_1(\theta_1) \equiv -45.16 \equiv 0,$$

$$P_2(\theta_1) \equiv 45.\frac{16.15}{1.2} - 40.8 \equiv -20,$$

$$P_3(\theta_1) \equiv -45.\frac{16.15.14}{1.\ 2.\ 3} \equiv 0,$$

$$P_4(\theta_1) \equiv 45.\frac{16.15.14.13}{1.\ 2.\ 3.\ 4} + 40.\frac{8.7}{1.2} - 36.4 \equiv 16,$$

etc., to $$P_{16}(\theta_1) \equiv R_{\bar{2}} + R_{\bar{3}} + R_{\bar{5}} \equiv 1 .$$

These relations, expressed by Theorem III, become

$$\Pi\,(x-\theta_1) \equiv R_{\bar{2}}\,(x-1)^{\tau_2(60)} + R_{\bar{3}}\,(x^3-1)^{\tau_3(60)} + R_{\bar{5}}\,(x^4-1)^{\tau_5(60)} \text{ mod. } 60 ,$$

or,

$$\Pi\,(x-\theta_1) \equiv 45\,(x-1)^{16} + 40\,(x^3-1)^{8} + 36\,(x^4-1)^{4} \text{ mod. } 60 .$$

II. $s = 5$.

$$P_m(\theta_5) \equiv (-)^{\bar{2}-1} R_{\bar{2}}\, C_{\frac{m}{\bar{2}-1}}^{\tau_{2.5}(60)} + (-)^{\frac{m}{\bar{3}-1}} R_{\bar{3}}\, C_{\frac{m}{\bar{3}-1}}^{\tau_{3.5}(60)} \text{ mod. } 60 .$$

$$P_1(\theta_5) \equiv -\,45\,.\,4 \equiv 0 ,$$

$$P_2(\theta_5) \equiv \quad 45\,.\,\frac{4\,.\,3}{1\,.\,2} - 40\,.\,2 \equiv 10 ,$$

$$P_3(\theta_5) \equiv -\,45\,.\,\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3} \equiv 0 ,$$

$$P_4(\theta_5) \equiv \quad 45 + 40 \equiv 25 = R_5 .$$

These relations, when expressed by Theorem III, become

$$\Pi\,(x-\theta_5) \equiv 45\,(x-1)^4 + 40\,(x^3-1)^2 + 36x^4 \text{ mod. } 60 .$$

III. $s = 3$.

$$P_m(\theta_3) \equiv (-)^{\frac{m}{\bar{2}-1}} R_{\bar{2}}\, C_{\frac{m}{\bar{2}-1}}^{\tau_{2.3}(60)} + (-)^{\frac{m}{\bar{5}-1}} R_{\bar{5}}\, C_{\frac{m}{\bar{5}-1}}^{\tau_{5.3}(60)} \text{ mod. } 60 ,$$

$$P_1(\theta_3) \equiv -\,45\,.\,8 \equiv 0 ,$$

$$P_2(\theta_3) \equiv \quad 45\,.\,\frac{8\,.\,7}{1\,.\,2} \equiv 0 ,$$

$$P_3(\theta_3) \equiv -\,45\,.\,\frac{8\,.\,7\,.\,6}{1\,.\,2\,.\,3} \equiv 0 ,$$

$$P_4(\theta_3) \equiv \quad 45\,.\,\frac{8\,.\,7\,.\,6\,.\,5}{1\,.\,2\,.\,3\,.\,4} - 36\,.\,2 \equiv 18 , \text{ etc., to}$$

$$P_8(\theta_3) \equiv \quad 45 + 36 \equiv 21 = R_3 ; \text{ otherwise,}$$

$$\Pi\,(x-\theta_3) \equiv \quad 45\,(x-1)^8 + 40x^8 + 36\,(x^4-1)^2 \text{ mod. } 60 .$$

IV. $s = 2$.

$$P_m(\theta_2) \equiv (-)^{\frac{m}{\bar{3}-1}} R_{\bar{3}}\, C_{\frac{m}{\bar{3}-1}}^{\tau_{3.2}(60)} + (-)^{\frac{m}{\bar{5}-1}} R_{\bar{5}}\, C_{\frac{m}{\bar{5}-1}}^{\tau_{5.2}(60)} \text{ mod. } 60 ,$$

$$P_1(\theta_2) \equiv 0 ,$$

$$P_2(\theta_2) \equiv -\,40\,.\,\frac{8\,.\,7}{1\,.\,2} \equiv 20 ,$$

$$P_3(\theta_2) \equiv 0 ,$$

$$P_4(\theta_2) \equiv 40\,.\,\frac{8\,.\,7\,.\,6\,.\,5}{1\,.\,2\,.\,3\,.\,4} - 36\,.\,4 \equiv 16 , \text{ etc., to}$$

$$P_8(\theta_2) \equiv 40 + 36 \equiv 16 = R_2 .$$

V. $s = 2.3$:

$$P_m(\theta_{2.3}) \equiv (-)^{\frac{m}{5-1}} R_{\bar{5}}\, C^{\tau_{5.2.3}(60)}_{\frac{m}{5-1}} \text{ mod. } 60\,,$$
$$P_1(\theta_{2.3}) \equiv P_2 \equiv P_3 \equiv 0\,,$$
$$P_4(\theta_{2.3}) \equiv -36.2 \equiv -12\,,$$
$$P_5 \equiv P_6 \equiv P_7 \equiv 0\,,$$
$$P_8(\theta_{2.3}) \equiv 36 = R_{2.3}\,.$$

VI. $s = 3.5$.

This is a case of exception (1), since $\dfrac{k}{\sigma} = \dfrac{60}{3.5} = 4$.

$$\therefore P_m(\theta_{3.5}) \equiv (-)^{\frac{m}{2}} R_{\bar{2}}\, C^{\frac{1}{2}\tau_{2.3.5}(60)}_{\frac{m}{2}} \text{ mod. } 60\,,$$
$$P_1 \equiv 0\,,$$
$$P_2 \equiv -45 = -R_{3.5}\,.$$

VII. $s = 5.2$.

$$P_m(\theta_{5.2}) \equiv (-)^{\frac{m}{3-1}} R_{\bar{3}}\, C^{\tau_{3.5.2}(60)}_{\frac{m}{3-1}} \text{ mod. } 60\,,$$
$$P_1 \equiv P_3 \equiv 0\,,$$
$$P_2(\theta_{5.2}) \equiv -40.2 = -20\,,$$
$$P_4(\theta_{5.2}) \equiv 40\,.\,\frac{2.1}{1.2} = R_{5.2}\,.$$

VIII. $s = 2.3.5$.

This is a case of exception (2), since $\dfrac{k}{\sigma} = \dfrac{60}{2^2.3.5} = 1$, and 60 contains 4.

$$\therefore P_1(\theta_{2.3.5}) \equiv \tfrac{1}{2}(60)\,, \text{ and } P_2 \equiv 0 = R_{2.3.5}\,.$$

The only two numbers included here are

$$0 \text{ and } 30\,,\; P_1 = 0 + 30\,, \text{ and } P_2 = 0.30\,.$$

Theorem V. If $k = a^t b^u \ldots q^s$, and $\Pi_k(x-\theta)$ denote the continued product, $x(x-1)(x-2)\ldots(x-\overline{k-1})$, then

$$\Pi_k(x-\theta) \equiv R_{\bar{a}}(x^a - x)^{\frac{k}{a}} + R_{\bar{b}}(x^b - x)^{\frac{k}{b}} + \ldots + R_{\bar{q}}(x^q - x)^{\frac{k}{q}} \text{ mod. } k\,;$$

except when one of the factors of k, as a, $= 2$, and $t > 1$, in which case

$$\Pi_k(x-\theta) \equiv R_{\bar{2}}\left[(x^2 - x)^2 - 2(x^2 - x)\right]^{\frac{k}{4}} + R_{\bar{b}}(x^b - x)^{\frac{k}{b}} + \ldots + R_{\bar{q}}(x^q - x)^{\frac{k}{q}} \text{ mod. } k.$$

To prove this theorem we have only to consider one component of the modulus, as a^t. Since $R_{\bar{b}}$, $R_{\bar{c}}$, ... $R_{\bar{q}}$ each contain a^t, and $R_{\bar{a}} \equiv 1$ mod. a^t, the congruence to be proved reduces to $\Pi_k(x-\theta) \equiv (x^a - x)^{\frac{k}{a}}$ mod. a^t. Now it is

evident that if we take the residues of the successive numbers less than k, we shall get $\frac{k}{a^t}$ successive groups. According to Theorem I, $\Pi(x-\theta)$ for one group $\equiv (x^a - x)^{a^{t-1}}$ mod. a^t. Therefore, $\Pi_k(x-\theta) \equiv (x^a - x)^{\frac{k}{a}}$ mod. a^t. But if $a=2$, and $t>1$, then $\Pi(x-\theta)$ for one group $\equiv [(x^2-1)(x^2-2x)]^{2^{t-2}}$ mod. 2^t, according to Theorem II. Therefore,

$$\Pi_k(x-\theta) \equiv [(x^2-1)(x^2-2x)]^{\frac{k}{4}} \equiv [(x^2-x)^2 - 2(x^2-x)]^{\frac{k}{4}} \text{ mod. } 2^t. \quad \text{Q. E. D.}$$

By comparing the coefficients of corresponding powers of x, we have the residues, mod. k, of the symmetric functions $\Sigma\alpha$, $\Sigma\alpha\beta$, &c. of the successive numbers from 0 to $k-1$ in terms of the repetents or residual units of k. When k does not contain 4, we may write the theorem more simply as follows, dispensing with the "carrier," x:

$$P_m(\theta) \equiv \sum_{\omega=a}^{\omega=q} (-)^{\frac{m}{\omega-1}} R_{\bar{\omega}} C^{\frac{k}{\omega}}_{\frac{m}{\omega-1}} \text{ mod. } k,$$

where C is a binomial coefficient, and where only those terms are to be included in the summation for which m is divisible by $\omega - 1$. But if $k = 2^t b^u c^v \ldots q^s$, and $t>2$, then it is easy to show that

$$P_m(\theta) \equiv (-)^m R_{\bar{2}} \left(C^{\frac{k}{2}}_m + \frac{k}{2} C^{\frac{k}{2}-1}_{m-2} + \frac{k}{2} C^{\frac{k}{2}-2}_{m-4} \right) + \sum_{\omega=b}^{\omega=q} (-)^{\frac{m}{\omega-1}} R_{\bar{\omega}} C^{\frac{k}{\omega}}_{\frac{m}{\omega-1}} \text{ mod. } k.$$

If $t=2$, the last term in the parenthesis is to be omitted; if $t=1$, the last two terms in the parenthesis are to be omitted, and the formula is then included under the preceding formula.

For example, suppose $k=30$, and it is required to find the residue mod. 30 of $P_4(\theta)$. We have $R_{\bar{2}} = 15$, $R_{\bar{3}} = 10$, and $R_{\bar{5}} = 6$. Then

$$P_4(\theta) \equiv (-)^4 R_{\bar{2}} C^{15}_4 + (-)^2 R_{\bar{3}} C^{10}_2 + (-)^1 R_{\bar{5}} C^6_1 \text{ mod. } 30,$$

or $$P_4(\theta) \equiv 15 \cdot \frac{15.14.13.12}{1.\ 2.\ 3.\ 4} + 10 \cdot \frac{10.9}{1.2} - 6 \cdot \frac{6}{1} \text{ mod. } 30,$$

or $$P_4(\theta) \equiv 15 + 0 - 6 \equiv 9 \text{ mod. } 30.$$

Since, now, the value of any symmetric function whatever can be obtained by means of the tables explicitly in terms of the symmetric functions $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, &c. we have thus a means of expressing, in terms of the repetents of k, the residues mod. k of any symmetric function of the s-totitives of k, or, finally, of any symmetric function of the successive numbers from 0 to $k-1$.

§ 2. *Extension of Preceding Results to the Theory of Functions (mod. p, $f(x)$).*

Let $f(x) = K \equiv A^t B^u \ldots Q^s$ mod. p, where p is a prime number, t, u, &c. are any integers, and A, B, &c. are irreducible (mod. p) functions of x, of

forms $x^a + \lambda_1 x^{a-1} + \lambda_2 x^{a-2} +$ &c., $x^b + \mu_1 x^{b-1} + \mu_2 x^{b-2} +$ &c., &c. respectively. It is well known that K can be so represented in only one way.

A function, $\phi(x)$, is said to contain another function, $f(x)$, *with respect to* p, when $\phi(x) = f(x)f_1(x) + pf_2(x)$. In the following the words *with respect to* p are always to be understood when the word *contain* is used with reference to functions, and when one function is said to be prime to another it is to be understood that they have no common factor *with respect to* p.

Let the number of incongruous (mod. p) functions of a less degree than that of K and prime to K be called the prime totient of K, and let it be denoted by $\tau_1(K)$ after the analogy of Professor Sylvester's notation and nomenclature in the case of integers. Likewise let the functions themselves be called the prime totitives of K. In the same way let the number of those which contain A but no other prime factor of K be called the A-totient of K, and let it be denoted by $\tau_A(K)$; and let the functions themselves be called the A-totitives of K. So on, for $\tau_{AB}(K)$, $\tau_{ABC}(K)$, &c. There are plainly 2^i different classes of totitives of K, if i denote the number of the unequal prime factors of K. A means of finding the value of the different totients of K will be furnished by the following:

Lemma. There are p^{n-m} incongruous (mod. p) functions of x, degree $< n$, which contain a given function $\phi(x)$ of form $x^m + \lambda_1 x^{m-1} + \lambda_2 x^{m-2} + \ldots + \lambda_m$.

For let $\psi(x) = \alpha x^{n-1} + \beta x^{n-2} +$ &c., where α, β, &c. are variable coefficients. Dividing $\psi(x)$ by $\phi(x)$, we get a remainder of degree $m - 1$. In order that there may be an exact division, each of the coefficients of the remainder must be equal to zero. We thus have m equations, which, as the process of division shows, are linear in the n quantities, α, β, &c. We may give arbitrary values to $n - m$ of these quantities, and considering the system of equations as a system of congruences mod. p, we may evidently satisfy the system in p^{n-m} different ways. Hence the *lemma* is proved.

Suppose, now, that $K = A^t B^u C^v$. We easily find, by means of the preceding lemma, and by use of the process employed to find the totients of an integer, that

$$\begin{aligned}
\tau_1(K) &= p^{(t-1)a} p^{(u-1)b} p^{(v-1)c} (p^a - 1)(p^b - 1)(p^c - 1), \\
\tau_A(K) &= \quad \text{"} \qquad (p^b - 1)(p^c - 1), \\
\tau_{AB}(K) &= \quad \text{"} \qquad (p^c - 1), \text{ \&c.}
\end{aligned}$$

The analogy between these numbers and the totients of an integer is at once apparent, viz: if an integer $k = a^t b^u c^v$ where a, b, c are prime numbers, the totients of K are derived from the totients of k by substituting in the latter

p^a for a, p^b for b, &c., where, however, a, b, c, the degrees of A, B, C, are not prime numbers, but any integers.

It is proposed, in what follows, to show briefly that certain theorems of my former paper, as well as those of the preceding section, have their analogues in the theory of functions (mod. p, K).

Let us first consider the properties of the roots of $X^2 \equiv X$ (mod. p, K), where $K = A^t B^u \ldots G^v H^w \ldots Q^z$. Let $s = H \ldots Q$, and $\sigma = H^w \ldots Q^z$. We evidently have $X \equiv 0$ (mod. p, σ), and $X \equiv 1$ $\left(\text{mod. } p, \frac{K}{\sigma}\right)$, and thence $\lambda\sigma - \mu \frac{K}{\sigma} \equiv 1$ mod. p. Since σ and $\frac{K}{\sigma}$ have no common factor with respect to p, this congruence gives one, and only one, value of λ. (See Serret, *Cours d'Alg. Sup.*, § 341), and consequently one, and only one, value of X for the two preceding congruences. This value contains σ. Call it R_s. It is evident now that there are twice as many roots of $X^2 \equiv X$ (mod. p, K) as there are ways of separating K into two factors prime to one another, viz: 2^i, and that one of them belongs to each of the 2^i classes of the totitives of K, where i is the number of the unequal prime factors in K. It is at once evident that these roots or repetents of K have the same properties as the integer roots of $x^2 \equiv x$ mod. k. For convenience, I restate some of them:

(1). $R_s R_{s'} \equiv R_{ss'}$ (mod. p, K), where s and s' are prime to each other.

(2). The sum of any given number of the repetents of K is congruous (mod. p, K) to the sum of the same number of any others of them, provided only that the product of the subscripts is the same for each sum.

(3). If $\bar{s}$ denote the product of all the unequal prime factors of K *not* contained in s, then

$$R_{\bar{s}} R_{\bar{s}'} \equiv 0 \text{ (mod. } p, K),$$

(4). $R_{\bar{s}} + R_{\bar{s}'} + \ldots + R_{\bar{s}^{[n]}} \equiv R_{\overline{ss' \ldots s^{[n]}}}$ (mod. p, K).

(5). If A', B', ... G' be prime totitives of A^t, B^u, ... G^v, respectively, and H', ... Q' are multiples of H, I, ... Q, respectively, then the residue (mod. p, K) of the function, $A'R_{\bar{A}} + B'R_{\bar{B}} + \ldots + Q'R_{\bar{Q}}$ is an s-totitive of K.

The analogue of Fermat's extended theorem is $X_s^{\tau_s(K)} \equiv R_s$ (mod. p, K), where X_s is an s-totitive of K.

If $K = A^t$, it is proved, in the ordinary way, that

$$X_1^{\tau_1(A^t)} \equiv 1 \text{ (mod. } p, K),$$

and it is, of course, obvious that

$$X_A^{\tau_A(A^t)} \equiv 0 \text{ (mod. } p, K).$$

Hence the theorem is true for $K = A^t$, since $R_1 = 1$ and $R_A = 0$ (mod. p, K). Then since [see (5)]

$$X_s \equiv A'R_{\bar{A}} + B'R_{\bar{B}} + \ldots + Q'R_{\bar{Q}} \text{ (mod. } p,\ K),$$

we have, by raising both sides to power $\tau_s(K)$, inasmuch as $\tau_s(K)$ contains $\tau_1(A^t), \ldots \tau_1(G^v)$,

$$X_s^{\tau_s(K)} \equiv R_{\bar{A}} + R_{\bar{B}} + \ldots + R_{\bar{G}} \equiv R_{\overline{AB\ldots G}} = R_s \text{ (mod. } p,\ K);$$

and the general theorem is proved.

Let it be required now to find the residues (mod. p, K) of the symmetric functions $\Sigma\Theta'_s$, $\Sigma\Theta'_s\Theta''_s$, &c., where Θ'_s, Θ''_s, &c. are the s-totitives of K, or, in other words, to find the residue of the function $\Pi_K(X - \Theta_s)$, in analogy with the results of the preceding section. First, let us prove that $\Pi_{A^t}(X - \Theta_1) \equiv [\Pi_{A^{t-1}}(X - \Theta_1)]^{p^a}$ (mod. p, A^t). The method of proof is precisely that of last section. Let α, β, γ, &c. be the prime totitives of A^{t-1}; then those of A^t will be given by $\alpha + \lambda A^{t-1}$, $\beta + \lambda A^t$, &c. where λ is any one of the p^a incongruous (mod. p) functions of a less degree than that of A. But

$$\Pi(X - \overline{\alpha + \lambda A^{t-1}}) \equiv X^{p^a} - (C_1^{p^a}\alpha + \Omega A^{t-1})X^{p^a-1} + (C_2^{p^a}\alpha^2 + C_1^{p^a}\Omega A^{t-1}\alpha)X^{p^a-2}$$
$$- (C_3^{p^a}\alpha^3 + C_2^{p^a-1}\Omega A^{t-1}\alpha^2)X^{p^a-3} + \text{\&c. (mod. } p,\ A^t)$$

just as before, where Ω here equals the sum of the p^a incongruous (mod. p) functions of a less degree than that of A. But we evidently have $\Omega \equiv 0$ mod. p.

$$\therefore \Pi(X - \overline{\alpha + \lambda A^{t-1}}) \equiv (X - \alpha)^{p^a}, \text{ (mod. } p,\ A^t).$$
$$\therefore \Pi_{A^t}(X - \Theta_1) \equiv [(X - \alpha)(X - \beta) \ldots]^{p^a}, \text{ (mod. } p,\ A^t).$$
$$\therefore \Pi_{A^t}(X - \Theta_1) \equiv [\Pi_{A^{t-1}}(X - \Theta_1)]^{p^a}, \text{ (mod. } p,\ A^t).$$

Now we know (*Serret*, § 345, *et seq.*) that

$$\Pi_A(X - \Theta_1) \equiv X^{p^a-1} - 1, \text{ (mod. } p,\ A),$$
$$\therefore \Pi_{A^2}(X - \Theta_1) \equiv (X^{p^a-1} - 1)^{p^a}, \text{ (mod. } p,\ A^2),$$
$$\therefore \Pi_{A^t}(X - \Theta_1) \equiv (X^{p^a-1} - 1)^{p^{a(t-1)}}, \text{ (mod. } p,\ A^t).$$

In the same way, we evidently get

$$\Pi_{A^t}(X - \Theta_A) \equiv (X)^{p^{a(t-1)}}, \text{ (mod. } p,\ A^t).$$

In general we have the following

Theorem. If $K = A^tB^u \ldots G^vH^w \ldots Q^r$, where A, B, &c. are different irreducible (mod. p) functions of x, of degrees, a, b, &c. and if $S = HI \ldots Q$, and Θ'_s, Θ''_s, &c. be the S-totitives of K, then

$$\Pi_K(X - \Theta_S) \equiv R_{\bar{A}}(X^{p^a-1} - 1)^{\tau_{AS}(K)} + \ldots + R_{\bar{G}}(X^{p^g-1} - 1)^{\tau_{GS}(K)}$$
$$+ R_{\bar{S}}X^{\tau_S(K)}, \text{ (mod. } p,\ K).$$

There are no exceptions to this formula as in the case of the corresponding formula for integers, for the unique prime number, 2, has no analogue in this theory of functions. The proof of the theorem is so nearly the same as that of the one for integers that it does not seem worth while to repeat it here. It is simply a substitution of p^a for a, p^b for b, &c. and (mod. p, K) for mod. k in the proof of Theorem III in the last section.

If $P_m(\Theta_s)$ denote $\Sigma\Theta'_s\Theta''_s\,..\,\Theta_{s'}^{[m]}$, we have as another form of the preceding theorem, analogous to Theorem IV of the last section,

$$P_m(\Theta_s) \equiv \sum_{\Omega=A}^{\Omega=G} (-)^{\frac{m}{\tau(\Omega)}} R_{\bar{\Omega}}\, C^{\frac{\tau_s(K)}{\tau(\Omega)}}_{\frac{m}{\tau(\Omega)}}, \text{ mod. } (p,\ K),$$

where the process of summation includes only those terms for which $\frac{m}{\tau(\Omega)}$, *i. e.*, $\frac{m}{p^\omega - 1}$ = an integer.

When $m = \tau_s(K)$, the formula becomes

$$P_{\tau_s(K)} \equiv R_{\bar{A}} + R_{\bar{B}} + \,..\, + R_{\bar{G}} \equiv R_{\overline{AB\,..\,G}} = R_S,$$

except when $\frac{K}{\sigma}$ = a power of an irreducible function, as A^t, and p is not 2, in which case $\frac{\tau_s(K)}{\tau(A)}$, *i. e.* $\frac{\tau_s(K)}{p^a - 1}$, is odd, and we have

$$P_{\tau_s(K)} \equiv -R_{\bar{A}} = -R_S, \text{ (mod. } p,\ K).$$

Since $P_{\tau_s(K)}$ = the product of the s-totitives of K, this special case of the above theorem constitutes the analogue of the generalized Wilsonian theorem.

It is easy to see that Theorem V of last section also has its analogue here, viz: If $\Pi_K(X - \Theta)$ denote the continued product $(X - \Theta')(X - \Theta'')\,...$ where Θ', Θ'', &c. are the whole set of the $\kappa = p^{a+b+\cdots+q}$ incongruous (mod. p, K) functions of x from 0 up to K, then

$$\Pi_K(X - \Theta) \equiv R_{\bar{A}}(X^{p^a} - X)^{\frac{\kappa}{p^a}} + \,..\, + R_{\bar{Q}}(X^{p^q} - X)^{\frac{\kappa}{p^q}}, \text{ [mod. } p,\ K],$$

or

$$P_m(\Theta) \equiv (-)^{\frac{m}{p^a-1}} R_{\bar{A}}\, C^{\frac{\kappa}{p^a}}_{\frac{m}{p^a-1}} + \,..\, + (-)^{\frac{m}{p^q-1}} R_{\bar{Q}}\, C^{\frac{\kappa}{p^q}}_{\frac{m}{p^q-1}}, \text{ (mod. } p,\ K).$$

BALTIMORE, 14 *April*, 1881.

Note on the Frequency of Use of the Different Digits in Natural Numbers.

By Simon Newcomb.

That the ten digits do not occur with equal frequency must be evident to any one making much use of logarithmic tables, and noticing how much faster the first pages wear out than the last ones. The first significant figure is oftener 1 than any other digit, and the frequency diminishes up to 9. The question naturally arises whether the reverse would be true of logarithms. That is, in a table of anti-logarithms, would the last part be more used than the first, or would every part be used equally? The law of frequency in the one case may be deduced from that in the other. The question we have to consider is, what is the probability that if a natural number be taken at random its first significant digit will be n, its second n', etc.

As natural numbers occur in nature, they are to be considered as the ratios of quantities. Therefore, instead of selecting a number at random, we must select two numbers, and inquire what is the probability that the first significant digit of their ratio is the digit n. To solve the problem we may form an indefinite number of such ratios, taken independently; and then must make the same inquiry respecting their quotients, and continue the process so as to find the limit towards which the probability approaches.

Let us suppose the numbers with which we commence to be arranged in periods according to the number of their digits, or, which is the same thing, according to the characteristics of their logarithms on the scale of which the basis is i, (i being 10 in the common system). Then, if two numbers are i^{c+s} and $i^{c'+s'}$, c and c' being integers, the significant figures of the ratio will be independent of c and c', since changing these integers will only change the decimal point. We may, therefore, take both numerator and denominator of the ratio out of the same period.

Moreover, since both numerator and denominator are formed by the same process, we may suppose the law of distribution of the numbers from which they are selected to be the same. Our problem is thus reduced to the following:

We have a series of numbers between 1 and i, represented by fractional powers of i, say i^s, the distribution of the exponents s, and therefore of the numbers, being according to any arbitrary law. Since these exponents are formed by casting off all the integers from a series of numbers, we may suppose them arranged around a circle according to some law. Then, if we select 2^n exponents at random and call them s', s'', s''', etc., the final ratio, obtained in the manner we have described, will be

$$i^{s' - s'' + s''' - s'''' + \text{etc.}}$$

The question is, what is the probability that the positive fractional portion of $s' - s'' + s''' - s'''' +$ etc., will be contained between the limits s and $s + ds$. It is evident that, whatever be the original law of arrangement, the fractions will approach to an equal distribution around the circle as n is increased, or the required probability will be equal to ds. But, the fractional part of $s' - s'' + s''' -$ etc. is the mantissa of the logarithm of the limiting ratio. We thus reach the conclusion:

The law of probability of the occurrence of numbers is such that all mantissæ of their logarithms are equally probable.

In other words, every part of a table of anti-logarithms is entered with equal frequency. We thus find the required probabilities of occurrence in the case of the first two significant digits of a natural number to be:

Dig.	First Digit.	Second Digit.
0	. . .	0.1197
1	0.3010	0.1139
2	0.1761	0.1088
3	0.1249	0.1043
4	0.0969	0.1003
5	0.0792	0.0967
6	0.0669	0.0934
7	0.0580	0.0904
8	0.0512	0.0876
9	0.0458	0.0850

In the case of the third figure the probability will be nearly the same for each digit, and for the fourth and following ones the difference will be inappreciable.

It is curious to remark that this law would enable us to decide whether a large collection of independent numerical results were composed of natural numbers or logarithms.

*Tables of the Generating Functions and Groundforms of the Binary Duodecimic, with some General Remarks, and Tables of the Irreducible Syzygies of certain Quantics.**

By J. J. Sylvester.

Generating Function for differentiants,

Denominator:

$$(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)$$
$$(1-a^{10})(1-a^{11}).$$

Numerator:

$$\begin{aligned}1 &+ 4a^2 + 17a^3 + 49a^4 + 125a^5 + 285a^6 + 594a^7 + 1143a^8 + 2063a^9\\ &+ 3517a^{10} + 5693a^{11} + 8817a^{12} + 13104a^{13} + 18769a^{14} + 25979a^{15} + 34830a^{16}\\ &+ 45317a^{17} + 57327a^{18} + 70595a^{19} + 84730a^{20} + 99214a^{21} + 113430a^{22}\\ &+ 126698a^{23} + 138345a^{24} + 147722a^{25} + 154297a^{26} + 157689a^{27} + 157689a^{28}\\ &+ 154297a^{29} + 147722a^{30} + 138345a^{31} + 126698a^{32} + 113430a^{33} + 99214a^{34}\\ &+ 84730a^{35} + 70595a^{36} + 57327a^{37} + 45317a^{38} + 34830a^{39} + 25979a^{40}\\ &+ 18769a^{41} + 13104a^{42} + 8817a^{43} + 5693a^{44} + 3517a^{45} + 2063a^{46} + 1143a^{47}\\ &+ 594a^{48} + 285a^{49} + 125a^{50} + 49a^{51} + 17a^{52} + 4a^{53} + a^{55}.\end{aligned}$$

Generating Function for covariants, reduced form,

Denominator:

$$(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})$$
$$(1-a^{11})(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)(1-ax^{10})(1-ax^{12}).$$

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* The tables of the duodecimic have been calculated by Mr. F. Franklin in accordance with Professor Sylvester's second method (see this Journal, Vol. III, p. 146), in pursuance of a grant made by the British Association for the Advancement of Science. The corresponding tables for the binary quantics of the first ten orders are given in this Journal, Vol. II, p. 223; those for systems of quantics of the first four orders, taken two and wo together, are given at page 293 of the same volume.

Numerator:*

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}
a^0	1																				
a^1		-1	-1	-1	-1	-1															
a^2			1	1	2	2	3	2	2	1	1										
a^3			1		1	-1	-1	-2	-3	-3	-3	-3	-2	-1	-1						
a^4	1		3	1	3		1			1	1	2	2	3	2	2	1	1			
a^5	1	1	4	2	2	-1	-2	-2	-3	-2	-2		-1		-1	-1	-1	-1	-1	-1	
a^6	3	3	7	4	4		-1	-2	-3			3	1	2	1	1					1
a^7	4	7	10	7	4	-1	-6	-7	-8	-4	-3	1	-1	1		-1	-1	-1			
a^8	7	12	17	10	6	-2	-11	-13	-12	-5	-2	5	8	6	2	1			1		
a^9	9	23	25	18	7	-6	-21	-24	-22	-10	-3	7	7	8	2	-1	-1	-2		-1	
a^{10}	17	36	39	25	5	-15	-39	-45	-37	-16	-1	16	16	15	5	1	-2	-3		-1	1
a^{11}	21	56	53	32	-1	-32	-67	-72	-54	-19	9	31	34	24	9	-1	-6	-7	-2	-1	1
a^{12}	36	81	76	44	-7	-52	-100	-108	-72	-17	24	56	56	35	11	-5	-14	-13	-4		3
a^{13}	45	112	97	51	-27	-93	-158	-162	-101	-21	45	85	87	47	11	-14	-24	-22	-6	3	4
a^{14}	65	151	133	63	-45	-134	-216	-218	-120	-3	90	143	136	69	15	-22	-40	-32	-8	7	7
a^{15}	81	199	163	66	-33	-206	-309	-303	-157	5	189	199	187	38	4	-48	-66	-51	-14	13	9
a^{16}	110	251	206	69	-124	-282	-404	-386	-173	52	226	301	267	114		-69	-97	-68	-16	24	17
a^{17}	131	309	241	59	-188	-389	-532	-489	-196	97	323	403	345	127	-29	-120	-146	-100	-23	37	21
a^{18}	168	370	288	51	-253	-495	-658	-580	-198	188	460	550	446	150	-58	-169	-202	-127	-28	57	36
a^{19}	193	483	318	22	-347	-636	-808	-692	-196	274	604	691	539	149	-116	-251	-274	-168	-26	33	45
a^{20}	232	493	359	-6	-436	-759	-939	-770	-153	421	797	883	655	160	-176	-329	-352	-204	-19	115	65
a^{21}	256	551	377	-54	-546	-912	-1093	-861	-119	554	980	1045	742	128	-277	-448	-450	-254	-18	154	81
a^{22}	293	598	402	-97	-648	-1035	-1209	-909	-35	746	1201	1252	844	110	-372	-553	-543	-292	-1	201	110
a^{23}	307	633	402	-161	-762	-1174	-1335	-956	-43	914	1402	1410	907	-45	-510	-697	-654	-343	14	251	131
a^{24}	336	667	407	-210	-852	-1266	-1404	-948	-168	1181	1619	1594	972	-14	-687	-821	-752	-375	44	308	168
a^{25}	339	679	381	-280	-953	-1371	-1480	-944	-274	1296	1791	1706	983	-119	-800	-972	-855	-417	69	369	193
a^{26}	351	678	367	-326	-1012	-1405	-1477	-867	-430	1508	1972	1836	997	-209	-934	-1086	-987	-434	114	430	232
a^{27}	339	664	323	-382	-1070	-1446	-1479	-810	-542	1632	2069	1860	987	-351	-1100	-1225	-1017	-457	148	490	256
a^{28}	336	635	291	-410	-1086	-1418	-1410	-692	-688	1782	2164	1900	898	-453	-1211	-1801	-1056	-446	208	547	293
a^{29}	307	595	239	-445	-1093	-1393	-1342	-595	-784	1837	2172	1837	784	-595	-1342	-1393	-1093	-445	239	595	307

* In the tabulated numerators, the *minus* sign is placed *over* the number which it affects.

Numerator—(*Continued*):

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}
a^{30}	293	547	203	$\overline{446}$	$\overline{1056}$	$\overline{1801}$	$\overline{1211}$	$\overline{453}$	893	1900	2164	1782	688	$\overline{692}$	$\overline{1410}$	$\overline{1418}$	$\overline{1086}$	$\overline{410}$	291	685	386
a^{31}	256	490	148	$\overline{457}$	$\overline{1017}$	$\overline{1225}$	$\overline{1100}$	$\overline{351}$	937	1860	2069	1682	542	$\overline{810}$	$\overline{1479}$	$\overline{1446}$	$\overline{1070}$	$\overline{382}$	328	664	339
a^{32}	232	430	114	$\overline{434}$	$\overline{987}$	$\overline{1086}$	$\overline{984}$	$\overline{209}$	997	1886	1972	1508	430	$\overline{867}$	$\overline{1477}$	$\overline{1405}$	$\overline{1012}$	$\overline{326}$	367	678	351
a^{33}	193	369	69	$\overline{417}$	$\overline{855}$	$\overline{972}$	$\overline{800}$	$\overline{119}$	983	1706	1791	1296	274	$\overline{944}$	$\overline{1480}$	$\overline{1371}$	$\overline{953}$	$\overline{280}$	381	679	339
a^{34}	168	308	44	$\overline{375}$	$\overline{752}$	$\overline{821}$	$\overline{687}$	$\overline{14}$	972	1594	1619	1131	168	$\overline{948}$	$\overline{1404}$	$\overline{1266}$	$\overline{852}$	$\overline{210}$	407	667	336
a^{35}	181	251	14	$\overline{343}$	$\overline{654}$	$\overline{697}$	$\overline{510}$	45	907	1410	1402	914	$\overline{43}$	$\overline{956}$	$\overline{1335}$	$\overline{1174}$	$\overline{762}$	$\overline{161}$	402	638	307
a^{36}	110	201	1	$\overline{292}$	$\overline{543}$	$\overline{553}$	$\overline{372}$	110	844	1252	1201	746	$\overline{85}$	$\overline{909}$	$\overline{1309}$	$\overline{1035}$	$\overline{648}$	$\overline{97}$	402	598	298
a^{37}	81	154	$\overline{18}$	$\overline{254}$	$\overline{450}$	$\overline{448}$	$\overline{277}$	128	742	1045	980	554	$\overline{119}$	$\overline{861}$	$\overline{1098}$	$\overline{912}$	$\overline{546}$	$\overline{54}$	377	551	256
a^{38}	65	115	$\overline{19}$	$\overline{204}$	$\overline{352}$	$\overline{329}$	$\overline{176}$	160	655	888	797	421	$\overline{153}$	$\overline{770}$	$\overline{939}$	$\overline{759}$	$\overline{436}$	$\overline{6}$	359	493	282
a^{39}	45	83	$\overline{26}$	$\overline{168}$	$\overline{274}$	$\overline{251}$	$\overline{116}$	149	539	691	604	274	$\overline{196}$	$\overline{692}$	$\overline{808}$	$\overline{636}$	$\overline{347}$	22	318	438	198
a^{40}	36	57	$\overline{23}$	$\overline{127}$	$\overline{202}$	$\overline{169}$	$\overline{58}$	150	446	550	460	188	$\overline{193}$	$\overline{580}$	$\overline{653}$	$\overline{495}$	$\overline{253}$	51	288	370	168
a^{41}	21	37	$\overline{23}$	$\overline{100}$	$\overline{146}$	$\overline{120}$	$\overline{29}$	127	345	403	323	97	$\overline{196}$	$\overline{489}$	$\overline{582}$	$\overline{389}$	$\overline{188}$	59	241	309	181
a^{42}	17	24	$\overline{16}$	$\overline{68}$	$\overline{97}$	$\overline{69}$		114	267	301	226	52	$\overline{178}$	$\overline{386}$	$\overline{404}$	$\overline{282}$	$\overline{124}$	69	206	251	110
a^{43}	9	18	$\overline{14}$	$\overline{51}$	$\overline{66}$	$\overline{48}$	4	83	187	199	189	5	$\overline{157}$	$\overline{303}$	$\overline{309}$	$\overline{206}$	$\overline{83}$	66	163	199	81
a^{44}	7	7	$\overline{8}$	$\overline{32}$	$\overline{40}$	$\overline{22}$	15	69	186	143	90	$\overline{3}$	$\overline{120}$	$\overline{218}$	$\overline{216}$	$\overline{134}$	$\overline{45}$	63	133	151	65
a^{45}	4	3	$\overline{6}$	$\overline{22}$	$\overline{24}$	$\overline{14}$	11	47	87	85	45	$\overline{21}$	$\overline{101}$	$\overline{162}$	$\overline{158}$	$\overline{93}$	$\overline{27}$	51	97	112	45
a^{46}	3		$\overline{4}$	$\overline{13}$	$\overline{14}$	$\overline{5}$	11	35	56	56	24	$\overline{17}$	$\overline{72}$	$\overline{108}$	$\overline{100}$	$\overline{52}$	$\overline{7}$	44	76	81	36
a^{47}	1	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{1}$	9	24	34	31	9	$\overline{19}$	$\overline{54}$	$\overline{72}$	$\overline{67}$	$\overline{32}$	$\overline{1}$	32	53	56	21
a^{48}	1	$\overline{1}$		$\overline{3}$	$\overline{2}$	1	5	15	16	16	$\overline{1}$	$\overline{16}$	$\overline{37}$	$\overline{45}$	$\overline{39}$	$\overline{15}$	5	25	39	36	17
a^{49}		$\overline{1}$		$\overline{2}$	$\overline{1}$	$\overline{1}$	2	8	7	7	$\overline{3}$	$\overline{10}$	$\overline{22}$	$\overline{24}$	$\overline{21}$	$\overline{6}$	7	18	25	28	9
a^{50}			1			1	2	6	3	5	$\overline{2}$	$\overline{5}$	$\overline{12}$	$\overline{18}$	$\overline{11}$	$\overline{2}$	6	10	17	12	7
a^{51}				$\overline{1}$	$\overline{1}$	$\overline{1}$		1	$\overline{1}$	1	$\overline{3}$	$\overline{4}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{1}$	4	7	10	7	4
a^{52}	1					1	1	2	1	3			$\overline{3}$	$\overline{2}$	$\overline{1}$		4	4	7	3	3
a^{53}		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$		$\overline{1}$		$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{2}$	$\overline{2}$	$\overline{1}$	2	2	4	1	1
a^{54}				1	1	2	2	3	2	2	1	1			1		3	1	3		1
a^{55}							$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	1		1		
a^{56}											1	1	2	2	3	2	2	1	1		
a^{57}																$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	
a^{58}																					1

Generating Function for covariants, representative form,

Denominator:

$$(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})\cdot(1-a^{11})(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})(1-a^2x^{16})(1-a^2x^{20})(1-ax^{12}).$$

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}
a^0	1																	
a^3			1	1	2	1	2	2	1	2	1	1	1	1		1		
a^4	1		3	2	4	3	4	4	3	4	2	3	1	2	1	1		1
a^5	1	2	5	6	7	8	6	9	5	6	3	4	1	1		$\overline{1}$		$\overline{2}$
a^6	3	4	9	11	13	15	13	15	9	11	5	6	2	1		$\overline{1}$		$\overline{3}$
a^7	4	10	16	21	23	27	23	24	18	15	10	6	3	$\overline{2}$	1	$\overline{5}$		$\overline{4}$
a^8	7	16	28	34	40	46	40	37	27	22	12	6		$\overline{5}$	$\overline{5}$	$\overline{10}$	$\overline{5}$	$\overline{7}$
a^9	9	30	44	58	64	71	64	55	39	27	13	$\overline{3}$	$\overline{8}$	$\overline{18}$	$\overline{16}$	$\overline{22}$	$\overline{14}$	$\overline{18}$
a^{10}	17	45	71	89	99	110	97	77	51	29	5	$\overline{15}$	$\overline{30}$	$\overline{40}$	$\overline{39}$	$\overline{42}$	$\overline{30}$	$\overline{23}$
a^{11}	21	78	106	133	148	156	137	101	62	20	$\overline{10}$	$\overline{50}$	$\overline{65}$	$\overline{88}$	$\overline{76}$	$\overline{74}$	$\overline{54}$	$\overline{38}$
a^{12}	36	102	153	191	208	218	187	123	61	1	$\overline{53}$	$\overline{102}$	$\overline{129}$	$\overline{146}$	$\overline{136}$	$\overline{120}$	$\overline{89}$	$\overline{59}$
a^{13}	45	148	214	265	287	288	240	148	55	$\overline{44}$	$\overline{113}$	$\overline{190}$	$\overline{220}$	$\overline{239}$	$\overline{218}$	$\overline{181}$	$\overline{181}$	$\overline{75}$
a^{14}	65	196	290	353	377	373	299	152	21	$\overline{114}$	$\overline{218}$	$\overline{309}$	$\overline{352}$	$\overline{363}$	$\overline{327}$	$\overline{258}$	$\overline{178}$	$\overline{89}$
a^{15}	81	264	379	460	486	460	357	147	$\overline{33}$	$\overline{226}$	$\overline{357}$	$\overline{488}$	$\overline{524}$	$\overline{528}$	$\overline{467}$	$\overline{344}$	$\overline{227}$	$\overline{90}$
a^{16}	110	332	486	577	601	558	408	118	$\overline{129}$	$\overline{376}$	$\overline{558}$	$\overline{694}$	$\overline{750}$	$\overline{725}$	$\overline{627}$	$\overline{489}$	$\overline{266}$	$\overline{68}$
a^{17}	131	419	602	707	728	647	442	61	$\overline{261}$	$\overline{587}$	$\overline{805}$	$\overline{972}$	$\overline{1018}$	$\overline{960}$	$\overline{810}$	$\overline{529}$	$\overline{286}$	$\overline{12}$
a^{18}	168	501	728	842	848	784	457	$\overline{80}$	$\overline{452}$	$\overline{840}$	$\overline{1122}$	$\overline{1287}$	$\overline{1330}$	$\overline{1216}$	$\overline{996}$	$\overline{604}$	$\overline{266}$	91
a^{19}	193	601	856	979	970	800	448	$\overline{160}$	$\overline{677}$	$\overline{1160}$	$\overline{1476}$	$\overline{1654}$	$\overline{1664}$	$\overline{1489}$	$\overline{1171}$	$\overline{644}$	$\overline{194}$	265
a^{20}	232	686	985	1106	1068	854	397	$\overline{331}$	$\overline{964}$	$\overline{1516}$	$\overline{1887}$	$\overline{2038}$	$\overline{2017}$	$\overline{1751}$	$\overline{1318}$	$\overline{686}$	$\overline{48}$	507
a^{21}	256	783	1102	1223	1158	867	318	$\overline{541}$	$\overline{1279}$	$\overline{1920}$	$\overline{2310}$	$\overline{2451}$	$\overline{2356}$	$\overline{1995}$	$\overline{1422}$	$\overline{556}$	171	842
a^{22}	293	854	1209	1319	1207	865	203	$\overline{785}$	$\overline{1635}$	$\overline{2332}$	$\overline{2758}$	$\overline{2831}$	$\overline{2673}$	$\overline{2183}$	$\overline{1451}$	$\overline{408}$	491	1259
a^{23}	307	931	1293	1388	1241	814	54	$\overline{1050}$	$\overline{1993}$	$\overline{2763}$	$\overline{3171}$	$\overline{3200}$	$\overline{2930}$	$\overline{2310}$	$\overline{1408}$	$\overline{151}$	898	1768
a^{24}	336	974	1352	1430	1225	744	$\overline{118}$	$\overline{1830}$	$\overline{2864}$	$\overline{3155}$	$\overline{3567}$	$\overline{3490}$	$\overline{3115}$	$\overline{2352}$	$\overline{1268}$	182	1898	2335
a^{25}	339	1015	1384	1437	1192	631	$\overline{310}$	$\overline{1607}$	$\overline{2691}$	$\overline{3525}$	$\overline{3876}$	$\overline{3719}$	$\overline{3204}$	$\overline{2299}$	$\overline{1032}$	614	1958	2967
a^{26}	351	1017	1385	1409	1110	507	$\overline{511}$	$\overline{1867}$	$\overline{2998}$	$\overline{3812}$	$\overline{4123}$	$\overline{3884}$	$\overline{3192}$	$\overline{2144}$	$\overline{698}$	1105	2575	3605
a^{27}	339	1015	1352	1353	1018	350	$\overline{704}$	$\overline{2095}$	$\overline{3224}$	$\overline{4088}$	$\overline{4249}$	$\overline{3860}$	$\overline{3067}$	$\overline{1898}$	$\overline{801}$	1661	3199	4244
a^{28}	336	974	1294	1267	887	203	$\overline{883}$	$\overline{2274}$	$\overline{3389}$	$\overline{4139}$	$\overline{4281}$	$\overline{3752}$	$\overline{2889}$	$\overline{1550}$	175	2223	3824	4823
a^{29}	307	931	1210	1156	762	88	$\overline{1082}$	$\overline{2399}$	$\overline{3452}$	$\overline{4162}$	$\overline{4183}$	$\overline{3558}$	$\overline{2518}$	$\overline{1143}$	658	2788	4374	5329
a^{30}	293	854	1105	1031	611	$\overline{100}$	$\overline{1146}$	$\overline{2456}$	$\overline{3446}$	$\overline{4059}$	$\overline{3998}$	$\overline{3251}$	$\overline{2127}$	$\overline{692}$	1158	3290	4850	5697
a^{31}	256	783	988	893	482	$\overline{236}$	$\overline{1219}$	$\overline{2447}$	$\overline{3333}$	$\overline{3876}$	$\overline{3701}$	$\overline{2881}$	$\overline{1690}$	$\overline{226}$	1612	3780	5188	5942

Numerator—(*Continued*):

x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}	x^{50}	x^{52}	x^{54}	x^{56}	x^{58}	x^{60}	x^{62}	x^{64}	x^{66}	x^{68}	x^{70}	
																		a^0
																		a^3
																		a^4
	$\overline{2}$		$\overline{1}$		$\overline{1}$													a^5
	$\overline{2}$		$\overline{1}$		$\overline{1}$													a^6
	$\overline{1}$		1		2		2		1		1							a^7
$\overline{2}$	$\overline{2}$	$\overline{1}$			2		1		1									a^8
$\overline{7}$	$\overline{4}$	$\overline{3}$	2		8		1				$\overline{1}$		$\overline{1}$		$\overline{1}$			a^9
$\overline{18}$	$\overline{8}$	$\overline{2}$	2	3	5	2	3	1			$\overline{1}$							a^{10}
$\overline{22}$	$\overline{6}$		10	7	11	6	7	2	1		$\overline{1}$		$\overline{1}$				1	a^{11}
$\overline{28}$	$\overline{8}$	11	20	20	22	15	11	4	1		$\overline{1}$	$\overline{1}$	$\overline{1}$					a^{12}
$\overline{32}$	15	31	45	40	43	29	18	7		$\overline{1}$	$\overline{5}$	$\overline{2}$	$\overline{8}$					a^{13}
$\overline{18}$	46	75	85	78	72	51	27	9	$\overline{1}$	$\overline{5}$	$\overline{9}$	$\overline{7}$	$\overline{4}$	$\overline{1}$				a^{14}
8	106	139	149	130	115	79	37	9	$\overline{10}$	$\overline{14}$	$\overline{19}$	$\overline{12}$	$\overline{8}$	$\overline{2}$			1	a^{15}
78	199	247	237	209	169	116	46	2	$\overline{28}$	$\overline{31}$	$\overline{34}$	$\overline{24}$	$\overline{18}$	$\overline{4}$	1	2	1	a^{16}
177	344	392	367	311	243	161	51	$\overline{11}$	$\overline{50}$	$\overline{58}$	$\overline{55}$	$\overline{37}$	$\overline{21}$	$\overline{5}$	3	4	3	a^{17}
348	540	602	526	445	326	206	48	$\overline{43}$	$\overline{90}$	$\overline{100}$	$\overline{88}$	$\overline{68}$	$\overline{28}$	$\overline{7}$	8	10	4	a^{18}
582	815	861	740	601	427	250	29	$\overline{94}$	$\overline{155}$	$\overline{168}$	$\overline{132}$	$\overline{89}$	$\overline{42}$	$\overline{7}$	14	16	7	a^{19}
906	1152	1192	980	780	525	279	$\overline{12}$	$\overline{179}$	$\overline{244}$	$\overline{250}$	$\overline{193}$	$\overline{132}$	$\overline{51}$	$\overline{5}$	27	30	9	a^{20}
1298	1574	1564	1264	962	621	287	$\overline{91}$	$\overline{298}$	$\overline{375}$	$\overline{365}$	$\overline{271}$	$\overline{174}$	$\overline{69}$	3	42	45	17	a^{21}
1788	2048	1998	1553	1149	692	260	$\overline{207}$	$\overline{468}$	$\overline{537}$	$\overline{510}$	$\overline{366}$	$\overline{232}$	$\overline{75}$	15	70	73	21	a^{22}
2334	2588	2443	1862	1312	784	191	$\overline{378}$	$\overline{680}$	$\overline{748}$	$\overline{684}$	$\overline{476}$	$\overline{283}$	$\overline{88}$	40	105	102	36	a^{23}
2947	3140	2910	2135	1441	728	57	$\overline{600}$	$\overline{954}$	$\overline{998}$	$\overline{890}$	$\overline{597}$	$\overline{348}$	$\overline{82}$	74	154	148	45	a^{24}
3568	3714	3338	2388	1510	656	$\overline{184}$	$\overline{886}$	$\overline{1269}$	$\overline{1285}$	$\overline{1116}$	$\overline{780}$	$\overline{394}$	$\overline{78}$	123	216	196	65	a^{25}
4201	4240	3733	2561	1511	513	$\overline{399}$	$\overline{1221}$	$\overline{1639}$	$\overline{1596}$	$\overline{1360}$	$\overline{860}$	$\overline{444}$	$\overline{48}$	189	293	264	81	a^{26}
4772	4721	4029	2667	1422	298	$\overline{723}$	$\overline{1617}$	$\overline{2029}$	$\overline{1942}$	$\overline{1605}$	$\overline{987}$	$\overline{466}$	$\overline{14}$	272	383	382	110	a^{27}
5285	5088	4239	2658	1250	$\overline{2}$	$\overline{1115}$	$\overline{2036}$	$\overline{2454}$	$\overline{2276}$	$\overline{1844}$	$\overline{1098}$	$\overline{480}$	53	370	490	419	131	a^{28}
5665	5349	4301	2561	981	$\overline{359}$	$\overline{1538}$	$\overline{2489}$	$\overline{2858}$	$\overline{2612}$	$\overline{2062}$	$\overline{1188}$	$\overline{454}$	124	488	606	501	168	a^{29}
5920	5442	4250	2331	684	$\overline{780}$	$\overline{2008}$	$\overline{2925}$	$\overline{3257}$	$\overline{2905}$	$\overline{2242}$	$\overline{1237}$	$\overline{417}$	233	615	731	601	193	a^{30}
6008	5403	4043	2023	225	$\overline{1231}$	$\overline{2446}$	$\overline{3347}$	$\overline{3592}$	$\overline{3161}$	$\overline{2374}$	$\overline{1251}$	$\overline{330}$	340	753	860	686	232	a^{31}

Numerator—(*Continued*):

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}
a^{32}	232	686	860	753	340	$\overline{330}$	$\overline{1251}$	$\overline{2374}$	$\overline{3161}$	$\overline{3592}$	$\overline{3347}$	$\overline{2446}$	$\overline{1281}$	225	2028	4048	5408	6003
a^{33}	193	601	731	615	233	$\overline{417}$	$\overline{1287}$	$\overline{2242}$	$\overline{2905}$	$\overline{3257}$	$\overline{2925}$	$\overline{2003}$	$\overline{780}$	634	2331	4250	5442	5920
a^{34}	168	501	606	488	124	$\overline{454}$	$\overline{1183}$	$\overline{2062}$	$\overline{2612}$	$\overline{2858}$	$\overline{2489}$	$\overline{1588}$	$\overline{359}$	981	2561	4301	5349	5665
a^{35}	131	419	490	370	53	$\overline{480}$	$\overline{1098}$	$\overline{1844}$	$\overline{2276}$	$\overline{2454}$	$\overline{2036}$	$\overline{1115}$	2	1250	2658	4239	5088	5285
a^{36}	110	332	383	272	$\overline{14}$	$\overline{466}$	$\overline{987}$	$\overline{1605}$	$\overline{1942}$	$\overline{2029}$	$\overline{1617}$	$\overline{723}$	298	1422	2667	4029	4721	4772
a^{37}	81	264	293	189	$\overline{48}$	$\overline{444}$	$\overline{860}$	$\overline{1360}$	$\overline{1596}$	$\overline{1689}$	$\overline{1221}$	$\overline{399}$	513	1511	2561	3733	4240	4201
a^{38}	65	196	216	123	$\overline{78}$	$\overline{394}$	$\overline{730}$	$\overline{1116}$	$\overline{1285}$	$\overline{1269}$	$\overline{886}$	$\overline{134}$	656	1510	2388	3338	3714	3568
a^{39}	45	148	154	74	$\overline{82}$	$\overline{348}$	$\overline{597}$	$\overline{890}$	$\overline{993}$	$\overline{954}$	$\overline{600}$	57	723	1441	2135	2910	3140	2947
a^{40}	36	102	105	40	$\overline{88}$	$\overline{283}$	$\overline{476}$	$\overline{684}$	$\overline{748}$	$\overline{680}$	$\overline{378}$	191	734	1312	1862	2443	2588	2334
a^{41}	21	73	70	15	$\overline{75}$	$\overline{232}$	$\overline{366}$	$\overline{510}$	$\overline{587}$	$\overline{468}$	$\overline{207}$	260	692	1149	1558	1998	2048	1788
a^{42}	17	45	42	3	$\overline{69}$	$\overline{174}$	$\overline{271}$	$\overline{365}$	$\overline{375}$	$\overline{298}$	$\overline{91}$	287	621	962	1264	1564	1574	1298
a^{43}	9	30	27	$\overline{5}$	$\overline{51}$	$\overline{132}$	$\overline{193}$	$\overline{250}$	$\overline{244}$	$\overline{179}$	$\overline{12}$	279	525	780	980	1192	1152	906
a^{44}	7	16	14	$\overline{7}$	$\overline{42}$	$\overline{89}$	$\overline{132}$	$\overline{163}$	$\overline{155}$	$\overline{94}$	29	250	427	601	740	861	815	582
a^{45}	4	10	8	$\overline{7}$	$\overline{28}$	$\overline{63}$	$\overline{88}$	$\overline{100}$	$\overline{90}$	$\overline{43}$	48	206	326	445	526	602	540	348
a^{46}	3	4	3	$\overline{5}$	$\overline{21}$	$\overline{37}$	$\overline{55}$	$\overline{58}$	$\overline{50}$	$\overline{11}$	51	161	243	311	367	392	344	177
a^{47}	1	2	1	$\overline{4}$	$\overline{18}$	$\overline{24}$	$\overline{34}$	$\overline{31}$	$\overline{23}$	2	46	116	169	209	237	247	199	73
a^{48}	1			$\overline{2}$	$\overline{8}$	$\overline{12}$	$\overline{19}$	$\overline{14}$	$\overline{10}$	9	37	79	115	130	149	139	106	8
a^{49}				$\overline{1}$	$\overline{4}$	$\overline{7}$	$\overline{9}$	$\overline{5}$	$\overline{1}$	9	27	51	72	78	85	75	46	$\overline{18}$
a^{50}					$\overline{3}$	$\overline{2}$	$\overline{5}$	$\overline{1}$		7	18	29	43	40	45	31	15	$\overline{32}$
a^{51}					$\overline{1}$	$\overline{1}$	$\overline{1}$		1	4	11	15	22	20	20	11	$\overline{3}$	$\overline{28}$
a^{52}	1				$\overline{1}$		$\overline{1}$		1	2	7	6	11	7	10		$\overline{6}$	$\overline{22}$
a^{53}							$\overline{1}$			1	3	2	5	3	2	$\overline{2}$	$\overline{3}$	$\overline{18}$
a^{54}			$\overline{1}$		$\overline{1}$		$\overline{1}$				1		3		2	$\overline{3}$	$\overline{4}$	$\overline{7}$
a^{55}									1		1		2			$\overline{1}$	$\overline{2}$	$\overline{2}$
a^{56}							1		1		2		2		1		$\overline{1}$	
a^{57}													$\overline{1}$		$\overline{1}$		$\overline{2}$	
a^{58}													$\overline{1}$		$\overline{1}$		$\overline{2}$	
a^{59}																		
a^{60}																		
a^{62}																		

Numerator—(*Continued*):

x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}	x^{50}	x^{52}	x^{54}	x^{56}	x^{58}	x^{60}	x^{62}	x^{64}	x^{66}	x^{68}	x^{70}	
5942	5188	3780	1612	226	1690	2881	3701	3876	3333	2447	1219	236	482	898	988	783	256	a^{32}
5697	4850	3290	1158	692	2127	3251	3998	4059	3446	2456	1146	100	611	1081	1105	854	293	a^{33}
5829	4374	2788	658	1148	2518	3558	4183	4162	3452	2399	1082	38	762	1156	1210	931	307	a^{34}
4828	3824	2223	175	1550	2839	3752	4281	4139	3889	2274	883	203	887	1267	1294	974	336	a^{35}
4244	3199	1661	301	1893	3067	3860	4249	4033	3224	2095	704	350	1018	1353	1352	1015	339	a^{36}
3605	2575	1105	698	2144	3192	3884	4123	3812	2998	1867	511	507	1110	1409	1385	1017	351	a^{37}
2967	1953	614	1032	2299	3204	3719	3876	3525	2691	1607	310	681	1192	1487	1384	1015	339	a^{38}
2335	1898	182	1263	2352	3115	3490	3567	3155	2364	1330	118	744	1225	1480	1352	974	336	a^{39}
1768	893	151	1408	2310	2980	3200	3171	2768	1993	1050	54	814	1241	1388	1298	931	307	a^{40}
1259	491	403	1451	2183	2678	2831	2758	2332	1635	785	203	865	1207	1319	1209	854	293	a^{41}
842	171	556	1422	1995	2356	2451	2310	1920	1279	541	318	867	1158	1223	1102	783	256	a^{42}
507	48	636	1318	1751	2017	2088	1887	1516	964	331	397	854	1068	1106	985	686	232	a^{43}
265	194	644	1171	1489	1664	1654	1476	1160	677	160	443	800	970	979	856	601	198	a^{44}
91	266	604	996	1216	1330	1287	1122	840	452	30	457	734	848	842	728	501	168	a^{45}
12	286	529	810	960	1018	972	805	587	261	61	442	647	728	707	602	419	131	a^{46}
68	266	439	627	725	750	694	558	376	129	118	408	558	601	577	486	332	110	a^{47}
90	227	344	467	528	524	483	357	226	33	147	357	460	486	460	379	264	81	a^{48}
89	178	258	327	368	352	309	218	114	21	152	299	373	377	358	290	196	65	a^{49}
75	131	181	218	239	220	190	113	44	55	143	240	288	287	265	214	148	45	a^{50}
59	89	120	136	146	129	102	58	1	61	123	187	218	208	191	158	102	36	a^{51}
38	54	74	76	83	65	50	10	20	62	101	137	156	148	133	106	78	21	a^{52}
23	30	42	39	40	30	15	5	29	51	77	97	110	99	89	71	45	17	a^{53}
18	14	22	16	18	8	3	13	27	39	55	64	71	64	58	44	30	9	a^{54}
7	5	10	5	5		6	12	22	27	37	40	46	40	34	28	16	7	a^{55}
4		5	1	2	3	6	10	15	18	24	28	27	23	21	16	10	4	a^{56}
3		1		1	2	6	5	11	9	15	18	15	18	11	9	4	3	a^{57}
2		1		1	1	4	3	6	5	9	6	8	7	6	5	2	1	a^{58}
1		1	1	2	1	3	2	4	3	4	4	3	4	2	3		1	a^{59}
		1		1	1	1	1	2	1	2	2	1	2	1	1			a^{60}
																	1	a^{63}

Table of Groundforms.

Degree in the Coefficients. \ Order in the Variables.	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	34
1							1										
2	1		1		1		1		1		1						
3	1		1	1	2	1	2	2	1	2	1	1	1	1		1	
4	2		3	2	4	3	4	4	3	4	2	3	1	2	1	1	1
5	2	2	5	6	7	8	6	9	5	6	3	4	1	1			
6	4	4	9	11	12	14	10	12	3	5							
7	5	10	15	20	18	21	9	8									
8	7	16	24	29	21	21											
9	9	28	33	37	15												
10	14	39	41	30													
11	15	53	40														
12	19	56	7														
13	18	44															
14	12																

The total number of groundforms (counting in the absolute constant and the quantic itself) is 949.

The manuscript sheets containing the original calculations from which the preceding tables have been constructed (as is the case also with the calculations connected with all the similar tables which have appeared in this journal) are deposited in the iron safe of the Johns Hopkins University, Baltimore, where they can be seen and examined, or copied, by any one interested in the subject. From the manifold independent systematic tests* to which the work has been

* One of these tests depends upon the following property of the generating function, which has been disclosed by observation, and of which the significance is not yet known. On putting $a=1$ in the numerator of the generating function, the coefficients of the various powers of x are integer multiples of the coefficient of x^0. Thus in the case of the duodecimic, the numerator of the reduced form becomes, on putting $a=1$,

$$5663\,(1+2x^2+x^4-x^6-3x^8-4x^{10}-4x^{12}-2x^{14}+2x^{16}+5x^{18}+6x^{20}+5x^{22}+2x^{24}-2x^{26}-4x^{28}-4x^{30}-3x^{32}\\ -x^{34}+x^{36}+2x^{38}+x^{40}).$$

Thus the numerical divisibility of the result of putting $a=1$ furnishes a test for the sums of the *columns*, while the algebraic divisibility of the result of putting $x=1$ (see this Journal, Vol. III, p. 151) tests the sums of the *rows*; and the satisfaction of both tests makes the correctness of the result practically certain.

subjected, Mr. Franklin estimates that the chance is far more than a million to one that the generating functions for the twelfthic as calculated do not contain a single numerical error. The highest order of any ground-covariant to the twelfthic it will be seen is 34, which is the superior limit of order given by M. Camille Jordan's formula for the ground-covariants to a system of an indefinite number of simultaneous binary forms of each of which the order is 12 or less: M. Jordan's "superior limit" in fact in this as in all the other calculated cases, being actually attained by one (and only one) ground-covariant to a single form.* It will also be noticed that for all orders of the primitive which have been calculated, viz., from 3 to 12 (with 11 omitted), the degree of the covariant of highest order is either 3 or 4. Looking at single quantics of the even orders 6, 8, 10, 12, it will be observed that the maximum order of their ground covariants for any degree (from and after the 4th degree) diminishes, or, to speak more strictly, never increases as the degree increases. As regards quantics of the odd orders 5, 7, 9, the same rule applies for the maximum order of their groundforms of even degrees; and in respect to their groundforms of odd degrees, the maximum order from and after the 3d degree diminishes or remains stationary as the degree increases. Also (alike for quantics of odd or even order) when (beginning with the 3d degree) in passing from an odd to the next even or from an even to the next odd degree of the groundforms, an increase in the maximum order takes place, it is only to the extent of a single unit. These facts, which constitute a sort of *law of shrinkage*, assume practical importance when the successive tables of groundforms are compared together, with a view to track the ground-differentiants, (or, in Mr. Cayley's language, the ground-seminvariants or *sources* of covariants) as the order of the primitive quantic is increased. Some of these ground-sources retain their irreducible character permanently, others only up to a particular limit of order in the primitive. The former may be regarded as the irreducible differentiants to a quantic of an infinite order: such for instance are all the differentiants of the second and third degree. But when we consider differentiants of the 4th degree this is no longer true. Thus we have the well-known example of the discrimi-

* It is also particularly noticeable that the number of the successively positive and negative blocks in the table follows the law observed in the inferior cases, viz. for Quantics of orders 3 and 4 there is a single block, for Quantics of orders 5 and 6 two blocks, for order 8 three blocks, and for orders 9 and 10 four blocks, there being five distinct blocks alternately positive and negative in the instance before us of the Quantic of order 12.

nant to $(a, b, c, d \between x, y)^3$, viz. $a^2d^2 + 4ac^3 + 4df^3 - 3b^2c^2 - 6abcd$, which is irreducible for this quantic, but for the quantic $(a, b, c, d, e \between x, y)^4$ remains, it is obvious, a differentiant, but no longer a ground-differentiant, being expressible under the form of the difference of two products of lower differentiants, viz., as

$$(ac - b^2)(ae - 4bd + 3c^2) - a \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

Suppose a differentiant to be the source of a covariant of the deg-order $j . \varepsilon$ considered as belonging to the quantic $(a_0, a_1, \ldots\ldots a_i \between x, y)^i$; then it is easily seen that it will be the source of a covariant of the deg-order $j . j + \varepsilon$ in respect to the quantic $(a_0, a_1, \ldots a_{i+j} \between x, y)^{i+1}$. We can, therefore, in many cases by a mere inspection of successive tables of groundforms eliminate some at least of the transient ground-differentiants: *i. e.* wherever there are K groundforms of deg-order $j . \varepsilon$ to a quantic of the order i, but only $K - \Delta$ of the deg-order $j . \varepsilon + \lambda j$ to the quantic of the order $i + \lambda$, we know that at least Δ of the sources to the K groundforms, *i. e.* Δ ground-differentiants of degree j and weight $\frac{ij - \varepsilon}{2}$ are only transiently irreducible. Thus *ex. gr.* the table of groundforms for the quintic exhibits a groundform of deg-order $4 . 4$, *i. e.* of deg-weight $4 . 8$; but the table of groundforms for the sextic contains no groundform of the same deg-weight, *i. e.* of deg-order $4 . 8$. Hence the differentiant of deg-weight $4 . 8$, although irreducible when regarded as a function of 6 letters (the number of letters which actually appear in it), is reducible when regarded as a function potentially of 7 or more.

So, again, for a like reason, the ground-differentiants of 5 letters, of deg-orders (in respect to the quintic) $5 . 1$ and $5 . 7$, *i. e.* of deg-weights $5 . 12$, $5 . 9$, are only transiently irreducible; and, what is very interesting, it will be seen at a glance (and here the law of shrinkage makes its importance felt) that the sources of all the groundforms to a quintic of a higher order than the 5th are only transitory (or provisional, so to say) ground-differentiants. So in like manner it will be recognized by comparing the tables of groundforms for the seventhic and eighthic, that of the 9 ground-sources of the degree 6 to the former, only two *can be* permanent, viz. one of the weight $\frac{6 . 7 - 2}{2}$ and one of the weight $\frac{6 . 7 - 4}{2}$, *i. e.* of the deg-weights $6 . 20$ and $6 . 19$ respectively: all

the others becoming resoluble when an additional letter is introduced into the quantic. Moreover, as the table for the eighthic contains no groundforms of deg-order 7.8, we see from the law of shrinkage that there can be no ground-source to the seventhic of a higher than the 6th degree which is permanently irreducible.*

A systematic weeding out of the transitory ground-sources from the published tables, which cannot in all cases for groundforms of earlier degrees be effected completely without an examination of a more searching kind than that illustrated by the above examples, must be reserved for a future occasion—after I shall have completed, as I hope soon to do, the study of a subject of higher interest and more pressing importance, which has for its object to determine not only the ground-forms so called, but also the ground-syzygants, the ground-counter-syzygants, etc., of quantics from their generating functions by a purely arithmetical process, which I believe to be already substantially in my possession.

As the first fruits of this method, I may state that the invariantive ground-syzygants (or, if the expression is preferred, fundamental syzygies) to the octavian quantic $(x, y)^8$ are 5 in number, and of the degrees 16, 17, 18, 19, 20 respectively in the coefficients. As regards the ground-syzygants (invariantive and covariantive) of the quintic, my method furnishes the same list as that given in Professor Cayley's Tenth Memoir on Quantics. Their deg-orders may be found as follows.

By the supernumerary ground-types understand the deg-orders of the ground-covariants exclusive of those represented by the factors which appear in the denominator of the representative generating function,† which are therefore $23 - 6$, *i. e.* 17 in number. Let these types be added each to itself and every other, thus giving rise to $\frac{17.18}{2}$ types: out of these sums strike out the types

$$8.4 \quad 9.5 \quad 10.2 \quad 10.4 \quad 11.3 \quad 12.2 \quad 14.4 \quad 16.2$$

and replace them by

$$13.5 \quad 14.6 \quad 15.3 \quad 15.5 \quad 16.4 \quad 17.3 \quad 19.5 \quad 21.3$$

The 153 types thus formed, together with the types, 26 in number, furnished by the negative terms in the numerator to the generating function, (see this Journal,

* For the 6th degree it will at once be seen that there can be no permanent differentiant to the seventhic except one of the 2d and one of the 4th order.

† In such denominator the number of factors for a Quantic of any odd order $2i-1$ is $3i-3$, and for any even order $2i$ is $3i-2$ (i in each case being supposed greater than unity).

vol. II, p. 224,) 179 in all, will be the deg-orders of the fundamental syzygants. Mr. Cayley founds this rule on his theory of the so-called Real Generating Function, which essentially consists in what may be termed the Dialytic Presentation of the Representative G. F. for the Quintic—namely as a sum of 26 pairs, each pair containing one positive and one negative term of the numerator divided by the denominator, so selected for conjunction that the developed expression of each pair shall be seen to be omni-positive by an obvious dialytic process.

The method followed by the eminent author in singling out the fundamental syzygants does not appear (as far as I can make out) to be explicitly stated in his memoir. The dialytic form (supposing, as is probably the case, it always exists for *finite* representative generating functions) is not easy to arrive at: a serious additional obstacle to the use of the dialytic method would arise in the case where (as for the seventhic) the numerator of the representative form becomes an infinite series. The method I employ does not require the use of the dialytic method, nor even of the *representative* form of the G. F., although the practical process is much simplified by the use of the representative form when it has a finite numerator. The result I obtain for the fundamental syzygants of the sextic is as follows: Take the 19 supernumerary ground-types (see vol. II, p. 225,) and add them each to each and to every other, as in the preceding case. Then strike out of the sums so formed the types of the deg-orders 6.4, 9.6, 8.4, 11.6, 10.4, 7.8, 8.6, 11.4, as well as one of the two sums 13.4 obtained from the addition of 5.2 and 8.2 or of 3.2 and 10.2 and replace the nine types so omitted by the eight types 12.8, 14.8, 13.6, 15.6, 10.10, 11.8, 14.6, 16.6. There will thus arise $19 \cdot \frac{20}{2} - 9 + 8$, or 189 types: to these adjoin the 29 types given by the negative terms in the numerator of the Rep. G. F.: the total number of types 189 + 29 or 218 so obtained will be the deg-orders of the complete system of fundamental syzygants to the sextic. The two types of the deg-order 6.6 which appear among the supernumerary types, it will of course be understood, are to be treated as distinct types in forming the binary sums. It is just barely possible (but I think very unlikely) that I may have committed some oversight in the table of replacement in the above calculation, and that the true number of ground-syzygies may be $19 \cdot \frac{18}{2} + 29$ or 219 instead of 218.*

* Nine binary sums of types are omitted, and are replaced by only eight other combinations. This is analogous to the loss of a unit in counting the irreducible syzygies to the invariants of an eighthic. The *supernumerary* invariants in this case are 3 in number; of degrees 8, 9, 10 respectively. Their binary combinations would give 6, but the true number of irreducible syzygies is only 5.

I subjoin a brief *aperçu* of the general theory.

A generating function (whatever its subject-matter) developed in a series consists of facients and coefficients, where any facient is a product of a finite set of letters each raised to a certain power. The totality of the exponents expressing these powers may be termed the type of the facient. In the generating functions to be referred to hereinunder, the letters employed are just as many in number as there are quantics in the system to be considered: viz., one letter corresponds to each quantic.

A generating function proper (with reference to the present theory) is defined to be one that is or can be developed into a series of facients whose coefficients and whose types are omni-positive integers, and where each such numerical coefficient is the number of linearly independent invariants whose degrees in the coefficients of the several quantics of the system are identical with the indices of the corresponding letters in the facient to which that numerical coefficient is attached.* The type of the facient may be also styled the type of the connoted invariants. A binomial expression consisting of unity followed by a facient and separated from it by the negative sign may be termed a *generator*.†

A proper generating function to a system of quantics may always by known methods (see this Journal, vol. III, p. 133) be expressed by a fraction whose numerator is a finite series of facients with numerical coefficients and its denominator a finite product of generators.

It may also be expressed (according to a definite process), and in one way only, by a fraction whose numerator and denominator alike consist of a finite or infinite (except in a few trivial cases, an infinite) product of generators.‡

* I speak designedly (for greater facility of expression) of invariants only, which can be done for binary quantics without any loss of generality, inasmuch as covariants may be regarded as invariants of a given system of quantics with a linear quantic superadded.

† If a, b, c, . . . are facients, $1-a^{\alpha}b^{\beta}c^{\gamma}$ . . . is a *generator*, and α, β, γ . . . (taken in a definite order) is its *type*.

‡ For instance let G be the generating function proper to the invariants of an eighthic.

$$\begin{aligned}
\text{Then } G &= \frac{1+a^8+a^9+a^{10}+a^{18}}{(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)} \\
&= [(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})]^{-1} \\
&\quad .\ (1-a^{16})(1-a^{17})(1-a^{18})(1-a^{19})(1-a^{20}) \\
&\quad .\ [(1-a^{25})(1-a^{26})(1-a^{27})(1-a^{28})(1-a^{29})]^{-1} \\
&\quad .\ (1-a^{33})(1-a^{34})(1-a^{35})^2(1-a^{36})^2(1-a^{37})^2(1-a^{38})(1-a^{39}) \\
&\quad .\ [(1-a^{41})(1-a^{42})^2(1-a^{43})^3(1-a^{44})^4(1-a^{45})^4(1-a^{46})^4(1-a^{47})^2(1-a^{48})^2]^{-1} \\
&\quad .\ \ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

A finite product of generators (or powers of generators) may be termed a generator-group.

For greater uniformity of statement in regard to what follows, let us agree to understand by a syzygant of the grade zero, an irreducible invariant. Then the two infinite products above referred to (whose ratio is algebraically equal to the generating function) may each be resolved into a product (usually infinite) of collect-groups, such that the totality of the types of the 1^{st}, 2^{d}, ... i^{th} groups of the denominator shall respectively represent the totality of the types of irreducible syzygants of the grades $0,\ 2, \ldots (2i-2)$ and the totality of the types of the 1^{st}, 2^{d}, ... i^{th} groups of the numerator the totality of the types of irreducible syzygants of the grades $1,\ 3,\ 5, \ldots (2i-1)$, so that each group may be said to be related to or to represent a complete system of irreducible syzygants of a certain grade (invariants being regarded as zero-graded syzygants)—that is to say, as many times as any generator is repeated in a group so many (and no more) irreducible syzygants of that type will there be of the corresponding grade.

Let G be a proper generating function to a system of quantics, $\Gamma_0,\ \Gamma_1,\ \Gamma_2 \ldots$ generator-groups such that

$$G = \frac{1 \,.\, \Gamma_1 \,.\, \Gamma_3 \,.\, \Gamma_5 \,.\,.}{\Gamma_0 \,.\, \Gamma_2 \,.\, \Gamma_4 \,.\, \Gamma_6 \ldots};$$

then, as suggested to me by Mr. Franklin, in order that the Γ series may be representative of complete systems of irreducible syzygants of the successive grades, it is *necessary* that $\frac{1}{\Gamma_0} - G$; $\frac{\Gamma_1}{\Gamma_0} - G$; $\frac{\Gamma_1\Gamma_3}{\Gamma_0} - G$; $\frac{\Gamma_1\Gamma_3}{\Gamma_0\Gamma_2} - G$; ... shall, when developed in series of facients with omni-positive indices, be alternately omni-positive and omni-negative. But the existence of these inequalities, although a *necessary*, is not a *sufficient* condition in order that the Γ's shall be so representative; *ex. gr.* $\Gamma_0 \,.\, \Gamma_2$ and $\Gamma_1 \,.\, \Gamma_3$ might evidently be regarded as single groups and the inequalities would still be satisfied; but suppose we further limit the Γ's in succession by the following rule, viz., that on withdrawing any one of the generator-factors from Γ_0 and calling Γ_0' the group so reduced $\frac{1}{\Gamma_0'} - G$ is no longer omni-positive, this will serve to define Γ_0 absolutely; Γ_0 being so determined, Γ_1 may in like manner be limited by the condition that its quotient by any one of its generators being called Γ_1', $\frac{\Gamma_1'}{\Gamma_0} - G$ shall be no longer omni-negative; then

Γ_1 is accurately determined, and, proceeding in like manner with each group in succession, the whole system of groups becomes exactly defined, and thus we obtain the necessary and sufficient condition of group-representation.

Calling $\frac{1}{\Gamma_0}, \frac{\Gamma_1}{\Gamma_0}, \frac{\Gamma_1\Gamma_3}{\Gamma_0}, \frac{\Gamma_1\Gamma_3}{\Gamma_0\Gamma_2}, \ldots \nu_0, \nu_1, \nu_2, \nu_3 \ldots$ respectively,

the ν series of quantities stand to G in somewhat the same relation as the complete quotients of a continued fraction to its complete value. Observe that $\nu_0 - 1, \nu_1 - 1, \nu_2 - 1, \ldots$ each vanish when the variables in G are each zero, and become infinite when the variables in G are each unity.

When each such variable has any value intermediate between 0 and 1, I think it almost certain that no two of the ν's can become equal, so that for all values of the variables inside those limits the parabolic lines or surfaces or hypersurfaces, &c., represented (after introducing a new variable ω) by the equations $\omega - \nu_0 = 0, \omega - \nu_1 = 0, \omega - \nu_2 = 0 \ldots$ (which coincide for the limiting values of the original variables at the origin and at a point at infinity) will never intersect, so that within the prescribed limits $\nu_0 - \nu_2, \nu_2 - \nu_4, \nu_4 - \nu_6 \ldots$ will be always positive and $\nu_1 - \nu_3, \nu_3 - \nu_5, \ldots$ will be always negative, the limited boundaries represented by

$$\omega - G, \ \omega - \nu_0, \ \omega - \nu_2, \ \omega - \nu_4, \ldots$$

being each external to the one that precedes it on one side of $\omega - G$, and

$$\omega - G, \ \omega - \nu_1, \ \omega - \nu_3, \ \omega - \nu_5, \ldots$$

following the same law on the other side. It is possible, moreover, that a more stringent condition than the above may be verified, viz., that

$$\begin{gathered} \nu_0 - G, \ \nu_2 - \nu_0, \ \nu_4 - \nu_2 \ldots \\ G - \nu_1, \nu_1 - \nu_3, \ \nu_3 - \nu_5 \ldots \end{gathered}$$

may each be developable into omni-negative functions, and again (to complete the analogy with the parallel theory of continued fractions or converging continued products) that

$$\nu_0 - G, \ G - \nu_1, \ \nu_2 - G, \ G - \nu_3, \ \nu_4 - G, \ldots$$

shall form a single series of continually decreasing quantities, or even in their developed state, of functions in which the corresponding coefficients to each facient form a continually decreasing (or, at least, never-increasing) series of

numbers. Then in the case of a single quantic, within the limits defined by the facient a being 0 and 1 the curves $\omega-\nu_1,\ \omega-\nu_3,\ \ldots\omega-G,\ \ldots\omega-\nu_2,\ \omega-\nu_0,$ will form an infinite series of loops having one common asymptote and one common point of intersection, and except at that one point keeping clear of each other.

I annex tables (pp. 58, 59) of the fundamental syzygants* (or if one pleases so to say) irreducible syzygies for the quintic and sextic, rendered more complete by inserting entries corresponding to the fundamental in- and- covariants. The positive integers correspond to these latter, the negative integers (the negative sign being set over the figure) to the irreducible syzygants. Thus *ex. gr.* in the table to the sextic the positive integer 2 found in the 6th line and 6th column, indicates that there are 2 ground-covariants of deg-order 6 . 6. The negative integer $\bar{7}$ found in the 12th line and 12th column indicates that there are 7 irreducible syzygies of deg-order 12 . 12.† The negative sign is appropriate, inasmuch as every independent syzygy of any deg-order lowers by a unit the number of linearly independent in- or- covariants of that deg-order that can be produced out of the inferior groundforms, so that syzygants may be regarded as negative existences in regard to groundforms: carrying on the same idea, counter-syzygants might be numbered by integers carrying two negative signs contradicting each other, and so on indefinitely.

The method of partitions or generating functions, which leads to these surprising constructions, looks at invariants and their connexions solely with regard to their deg-order or type without taking any account of their content; in other words it deals only with the *idea* or *notion* of these beings and their relations,

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* N. B.—A syzygant to a Quantic is a rational integer function of its in- or- covariants which, expressed as a function of the coefficients, vanishes identically, but we may still understand its "degree in the coefficients" to mean the degree of any one of the terms of which it is the sum.

† If j or e exceed the highest degree or order respectively found in any table, or, if without that being the case there is a blank space in the j^{th} line and e^{th} column of the table, the meaning is that there is no irreducible groundform or syzygy of the deg-order $j.e$. In the tables exhibited it will be seen that the deg-order $j'.e'$ of each syzygant is superior to the deg-order $j.e$ of every groundform: *i. e.* the differences $j'-j$, $e'-e$ are neither of them less and one of them is greater than zero. The same is true for all quantics which have a finite Rep. G. F., but not necessarily and probably never actually so in other cases; thus *ex. gr.* to the seventhic belongs an irreducible invariant of degree 22 and an irreducible syzygy of degree 20, so that here the $j'.e'$ (20.0) is inferior to the $j.e$ (22.0). The fact of every $j'.e'$ being superior to the $j.e$ can be expressed by saying that the invariantive syzygetic portions of a Rep. G. F. table are not intermingled but lie totally apart and may be divided from each other by a single continuous cut.

and may therefore, I think, suitably be termed the Idealistic method.* I cannot see the faintest possibility of the symbolic method serving to determine a complete system of syzygies in any but the trivial cases of quantics of the 3d or 4th order—the only cases where the infinite procession of beings (syzygants, counter-syzygants, anti-counter-syzygants, etc.,) rising out of each other, comes to a stop—there being for those cases no procession after the 1st step, as is also true of invariants (as distinguished from covariants) for quantics of the 6th order. This is how it came to pass in the infancy of the theory that the number of ground-covariants was supposed to become infinite for quantics beyond the fourth and their ground-invariants for quantics beyond the 6th order.

I think it may be interesting to some of the readers of the Journal to be put in possession of the complete system of irreducible syzygies to a system of two or more quantics, and I select as an easy example the case of a combined quadratic and cubic, reserving the other combinations of which the groundform tables have been published for a subsequent number of the Journal. The supernumerary groundforms for the quadri-cubic system (see this Journal, vol. II, pp. 295, 296,) are of the deg-deg-orders $3.4.0$, $1.1.1$, $2.1.1$, $1.3.1$, $2.3.1$, $1.2.2$, $1.1.3$, $0.3.3$, where the first and second numbers express the degrees in the coefficients of the quadric and cubic respectively, and the last number

* My proof in the *Phil. Trans.*, founded on the canonical form of the Quintic, of its 4th, 8th, 12th and 18th-degreed invariants forming a complete system, the late Mr. Boole's discovery of the cubinvariant to the Quartic, the various disproofs in the *Comptes Rendus* and in this Journal of the existence of supposed groundforms, are all exemplifications of the Realistic point of view. The symbolic lies between this and the Idealistic aspect of the subject, in so far as the operations by which invariants are engendered constitute a new and so to say finer subject-matter, capable of being itself operated upon in all respects like ordinary algebraical substance. In Professor Cayley's 10th Memoir on Quantics there is a sort of half return from the Idealistic to the Realistic view—a kind of substantiality being attributed to the groundforms themselves as primary elements in the study of their syzygetic interconnections. It may be well to notice, for the benefit of the readers of that memoir (*Phil. Trans.* 1878,) that in the Representative Form given at p. 657 two terms are omitted by an oversight, viz. $-a^{17}x^4$ and a^3x^{12}. I need hardly add (since the publication of my tables in this Journal), with reference to a doubt expressed by Prof. Cayley (*loc. cit.*), that I *had* obtained the form referred to in the paragraph following the R. G. F. in question, though *not* by dividing out the common factors from the numerator and denominator of the R. G. F.; on the contrary, the N. G. F. is first obtained from the generating function in its crude form (which if left in that form would lead to a bivergent series), and then the R. G. F. is obtained from this, through multiplying its numerator and denominator by the factors needed to render the denominator a product of representative groundforms.

The Symbolic and the Idealistic (which I formerly called the fatalistic or peprotic) method alike, as far as is known, owe their conception to the same (unnecessary to be named) acute and capacious intellect. Whether very much that is essential, remains to be added to the great discoveries of Gordan and Jordan in the direction of the former may reasonably be doubted, but no such misgiving can be entertained with respect to the latter, which already has given rise to many more questions than it has settled (of a kind, too, of which a solution sooner or later may reasonably be anticipated).

ORDER IN THE VARIABLES.

DEGREE IN THE COEFFICIENTS.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1						1													
2			1				1												
3				1		1				1									
4	1				1		1												
5		1		1				1				$\bar{1}$							
6			1		1		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{2}$				$\bar{1}$
7		1				1		$\bar{1}$		$\bar{3}$		$\bar{1}$		$\bar{2}$		$\bar{2}$			
8	1		1				$\bar{2}$		$\bar{2}$		$\bar{3}$		$\bar{3}$				$\bar{1}$		
9				1		$\bar{1}$		$\bar{5}$		$\bar{2}$		$\bar{2}$		$\bar{3}$					
10					$\bar{1}$		$\bar{3}$		$\bar{4}$		$\bar{5}$				$\bar{2}$				
11		1				$\bar{4}$		$\bar{3}$		$\bar{3}$		$\bar{3}$							
12	1				$\bar{2}$		$\bar{4}$		$\bar{5}$		$\bar{1}$		$\bar{2}$						
13		1		$\bar{2}$		$\bar{3}$		$\bar{3}$		$\bar{4}$									
14			$\bar{1}$		$\bar{2}$		$\bar{6}$		$\bar{1}$		$\bar{3}$								
15				$\bar{2}$		$\bar{3}$		$\bar{4}$		$\bar{1}$									
16					$\bar{5}$		$\bar{2}$		$\bar{2}$		$\bar{2}$								
17				$\bar{2}$		$\bar{3}$		$\bar{2}$		$\bar{1}$									
18	1		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{1}$								
19				$\bar{2}$		$\bar{3}$		$\bar{1}$											
20			$\bar{2}$		$\bar{1}$		$\bar{2}$		$\bar{1}$										
21				$\bar{3}$		$\bar{1}$				$\bar{1}$									
22			$\bar{1}$		$\bar{2}$		$\bar{1}$												
23		$\bar{1}$		$\bar{1}$				$\bar{1}$				$\bar{1}$							
24			$\bar{2}$		$\bar{1}$														
25		$\bar{1}$				$\bar{1}$													
26			$\bar{2}$																
27				$\bar{1}$															
28																			
29		$\bar{1}$																	
30																			
31		$\bar{1}$																	
32																			
33																			
34																			
35																			
36	$\bar{1}$																		

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

ORDER IN THE VARIABLES.

DEGREE IN THE COEFFICIENTS.

Table of Groundforms and Irreducible Syzygies to the Sextic.

Degree in the Coefficients \ Order in the Variables	0	2	4	6	8	10	12	14	16	18	20	22	24
1				1									
2	1		1		1								
3		1		1	1		1						
4	1		1	1		1							
5		1	1		1				$\bar{1}$				
6	1			2	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$		$\bar{1}$
7		1	1	$\bar{1}$		$\bar{1}$	$\bar{4}$	$\bar{1}$	$\bar{2}$	$\bar{2}$		$\bar{1}$	
8		1			$\bar{2}$	$\bar{4}$	$\bar{2}$	$\bar{4}$	$\bar{8}$		$\bar{2}$		
9			1		$\bar{4}$	$\bar{2}$	$\bar{5}$	$\bar{4}$		$\bar{8}$			
10	1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{6}$	$\bar{5}$	$\bar{2}$	$\bar{4}$				
11					$\bar{6}$	$\bar{5}$	$\bar{2}$	$\bar{4}$					
12		1	$\bar{1}$	$\bar{4}$	$\bar{8}$	$\bar{8}$	$\bar{7}$	$\bar{1}$	$\bar{1}$				
13			$\bar{1}$	$\bar{2}$	$\bar{5}$	$\bar{5}$	$\bar{1}$	$\bar{8}$					
14			$\bar{1}$	$\bar{4}$	$\bar{6}$	$\bar{2}$	$\bar{3}$						
15	1		$\bar{8}$	$\bar{3}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{2}$					
16			$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{1}$	$\bar{2}$						
17			$\bar{8}$	$\bar{8}$	$\bar{1}$	$\bar{2}$		$\bar{1}$					
18		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{4}$		$\bar{1}$						
19			$\bar{2}$	$\bar{3}$		$\bar{1}$							
20		$\bar{1}$	$\bar{8}$		$\bar{1}$				$\bar{1}$				
21				$\bar{3}$									
22		$\bar{1}$	$\bar{2}$										
23		$\bar{1}$											
24			$\bar{2}$										
25		$\bar{1}$											
26													
27		$\bar{1}$											
28													
29													
30	$\bar{1}$												

expresses the order in the variables. Adding each of these triads to itself and every other, rejecting the combinations 2.2.2, 3.2.2, 2.4.2, which appear in the numerator of the G. F. (and arise from the additions 1.1.1 + 1.1.1, 1.1.1 + 2.1.1, 1.1.1 + 1.3.1,) replacing them by the higher combinations 1.1.1 + 1.1.1 + 1.1.1, 1.1.1 + 1.1.1 + 2.1.1, 1.1.1 + 1.1.1 + 1.3.1, *i.e.* 3.3.3, 4.3.3, 3.5.3, and adding in the 12 types furnished by the negative terms in the numerator of the G. F., the totality of the irreducible syzygies (48 in number) to the binary quadri-cubic system is arrived at and exhibited in the annexed table, in which the exponents attached to any type signify the number of irreducible syzygies of the corresponding deg-deg-order.

Table of Irreducible Syzygies to the Quadri-cubic System.

6.8.0,	4.5.1,	4.7.1,	5.5.1,	5.7.1,	2.6.2,	$(3.4.2)^2$,
$(3.6.2)^2$,	4.2.2,	$(4.4.2)^2$,	$(4.6.2)^2$,	1.5.3,	2.3.3,	$(2.6.3)^2$,
$(3.3.3)^2$,	$(3.5.3)^3$,	3.6.3,	3.7.3,	4.3.3,	4.5.3,	1.4.4,
1.6.4,	2.2.4,	$(2.4.4)^4$,	2.6.4,	3.2.4,	3.4.4,	3.6.4,
$(1.5.5)^2$,	2.3.5,	2.5.5,	3.5.5,	0.6.6,	1.3.6,	1.4.6,
2.2.6,	4.7.6,					

there being thus one irreducible invariantive syzygy and 4, 10, 12, 11, 5, 5 covariantive syzygies of orders 1, 2, 3, 4, 5, 6 respectively.

It may be worth while just to notice that the types to the complete system of irreducible syzygies to a simultaneous linear and quartic form will consist simply of the sums of the 13 supernumerary types, (A. M. J., vol. II, p. 295,) 6.3.0, 3.1.1, 3.2.1, 5.3.1, 2.1.2, 2.2.2, 4.3.2, 1.1.3, 1.2.3, 3.3.3, 2.3.4, 1.3.5, 0.3.6, added each to itself and every other, together with the 14 types taken from the negative terms in the numerator of the G. F., viz: 7.3.1, 6.3.2, 5.3.3, 4.3.4, 6.4.4, 6.5.4, 3.3.5, 5.4.5, 5.5.5, 2.3.6, 4.4.6, 4.5.6, 1.3.7, 7.6.7, making $\frac{13.14}{2} + 14$, *i.e.* 105 in all. In this instance there is no rejection or substitution of sums called for.

A word or two seems necessary to leave unambiguous the meaning of the term syzygants of any specified grade in what precedes.

In- or- covariants may be termed syzygants of grade zero (as already stated). Syzygants of the first grade are defined to be rational integer functions of those of grade zero which vanish when the latter are expressed in terms of the original coefficients. It is not *necessary* to define *these* syzygants as functions of *irreducible* ones of grade zero (which vanish under the condition aforesaid), because every in- or- covariant is a rational integer function of the irreducible in- or- covariants. But when we come to syzygants of the second grade (since those of the first grade are not necessarily functions of the irreducible ones of that grade, but may be so of the in- or- covariants as well), it becomes necessary to define syzygants of the second grade (*aliter* counter-syzygants) as rational integer functions of *irreducible* ones of the first grade which vanish when they are expressed in terms of the quantities (here the in- or- covariants) which immediately precede them in the scale of generation. And so, in general, following out the defining process step by step, by a syzygant of the $(i+1)^{\text{th}}$ grade for the purpose of this theory, is to be understood a rational integer function of the *irreducible* ones of the i^{th} grade which vanishes when these latter are expressed in terms of those of the grade $i-1$. Such at least is my present impression; but, supposing that I am laboring under a misconception on this point, it will in nowise affect the validity of the theory in what regards the computation of the irreducible in- or- covariants and the syzygants of the first grade.

A Demonstration of the Impossibility of the Binary Octavic possessing any Groundform of deg-order 10.4.

By J. J. Sylvester.

Dr. von Gall has rendered an inestimable service to algebraical science by working out, according to Gordan's method, the complete system of groundforms to the octavian binary quantic $[(x, y)^8]$. His results, published in the *Mathematische Annalen*, were at first widely discordant from those which have appeared in this Journal, but eventually have been brought by their author into perfect agreement with them, with the sole exception that his table includes a covariant of *deg-order* 10.4, not included in my list, which he states that he has not been able to decompose: it is the object of the present communication to bring the two tables into exact accord by demonstrating that no irreducible covariant to $(x, y)^8$ of that deg-order can exist. The total number of covariants of deg-order 10.4 obtained by multiplying together the irreducible covariants of an inferior deg-order (which appear equally in von Gall's table and in my own, and whose existence therefore may be taken for granted)* will be seen to be 32, which is the number of linearly independent covariants of that deg-order given by Cayley's law, (see p. 80); hence, by the fundamental postulate, the 32 compounds in question must not be supposed subject to any linear relation, so that, according to that postulate, there exists no groundform of the deg-order in question; but my object is to use this instance as another exemplification of the validity of that

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* As I have elsewhere remarked, since no groundforms can exist exterior to the tables furnished by Gordan's method, and no reducible forms can be contained in the tables furnished by the English method, it follows, even without assuming the truth of the fundamental postulate, that wherever the tables furnished by the two methods accord, they must, of logical necessity, be correct, *mere errors of calculation excepted.*

same very reasonable postulate—as I have done on the three former occasions where the tables of Clebsch, Gordan and Gundelfinger comprised groundforms extraneous to the tables obtained by me on the assumption of its truth; the proof, however, on the present occasion, is much lengthier than any that has ever hitherto been employed, and involves arithmetical computations of considerable prolixity, all necessity for which I had, in previous cases, been able to evade. It is, I may add, only after repeated trials and discomfitures, that I have succeeded at length in devising a special method adequate to prove the important point at issue.

The irreducible invariants and covariants of deg-order *inferior* to 10.4, (*i. e.* whose degree in the coefficients and whose order in the variables are not each as great as 10 and 4 respectively,) and which also can enter as factors of a covariant of deg-order 10.4 (this excludes the necessity of considering invariants of degrees 9 or 10) are as follows: the invariants are of degrees 2, 3, 4, 5, 6, 7, 8, one of each degree; the covariants are one of deg-orders 5.2, 2.4, 3.4 respectively, and two of deg-orders 4.4, 5.4, 6.4, 7.4, 8.4 respectively. We may denote the invariants by 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, and the covariants by 5.2, 2.4, 3.4, 4.4, 4.4^*, 5.4, 5.4^*, 6.4, 6.4^*, 7.4, 7.4^*, 8.4, 8.4^*, and it is an easy arithmetical calculation to show (see *Comptes Rendus*, July 25, 1881) that there are (as already stated) 32 different ways in which these duads, by their combination, can give rise to the duad 10.4. Out of these 32 it is important, with a view to what follows, to isolate those in which neither 2.0 nor 3.0 appears; their number will easily be seen to be 10, as shown in the scheme below—

$4.0+6.4$	$4.0+6.4^*$	$4.0+4.0+2.4$	$5.0+5.4$	$5.0+5.4^*$
$6.0+4.4$	$6.0+4.4^*$	$7.0+3.4$	$8.0+2.4$	$5.2+5.2$.

What I have to prove is, that no equation $\Omega=0$ exists, where Ω is a linear function of the 32 products in question, connected by numerical coefficients. Suppose it can be shown that Ω does not contain any of the 10 functions above indicated. Then Ω is either of the form $(2.0)U$ or $(3.0)V$, or is a linear function of $(2.0)U$ and $(3.0)V$. In the former two cases we should obtain $U=0$ or $V=0$ respectively; and in the third case the equation $\lambda(2.0)U+\mu(3.0)V=0$, since 2.0 and 3.0 have no common factor, implies the existence of an integral equation $\lambda\dfrac{U}{(3.0)}+\mu\dfrac{V}{(2.0)}=0$. Hence, in the

three cases supposed, there would exist a syzygy of the deg-order 8.4, 7.4, 5.4 respectively between composite covariants of the inferior deg-orders; but if this were so, the number of irreducible covariants of one or another of these deg-orders would not be what it is at present, but, in order to satisfy Cayley's law, would have to be increased by a unit: or, in other words, results obtained by my method and coincident with those resulting, or capable of resulting, from the German method, would be erroneous, which never can be the case.* Hence, the non-existence of $\Omega = 0$ will be demonstrated if it can be shown that, for some particular form of the general primitive $(x, y)^8$ which causes the invariants of the second and third degrees each to vanish, the particular values then assumed by the 10 compounds which remain in Ω are not subject to any linear relation. Of course the converse would not be true; the fact of the existence of a syzygy between these 10, or even between the whole 32 compounds for a special form of the primitive, would not establish the existence of a syzygy between them in the general case.

The great practical gain of making the first two invariants vanish is that it leads to a computation in which only 10 instead of 32 linear functions have to be handled—but it is not possible *à priori* before the calculations have been gone through, to feel at all assured that the particular form assumed may not be such as to lead only to nugatory results. Such happily, however, turns out not to be the case with the form I am about to employ which leads to the expression of the 10 compounds as homogeneous linear functions of 11 arguments,† giving rise to a rectangular matrix 11 places wide and 10 places deep of which it can be shown that the complete minors (determinants of the 10th order) do not all vanish, so that the 10 functions cannot be subject to any syzygy; and consequently, if $\Omega = 0$ were a really existing syzygy, Ω must consist exclusively of 22 terms, every one of which contains one or both of the two first invariants; but this has been shown to be impossible, so that the non-evanescence of the minors referred to at once establishes the non-existence of a syzygy of deg-order 10.4, and, therefore, the non-existence of a *ground-form of that deg-order.*

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*Towards the end of this paper I establish the same conclusion by a more direct method, in which nothing extraneous to Dr. von Gall's own table is assumed, except the one fact of the linearly independent covariants of deg-order 10.4 being 32 in number.

† One of these arguments is itself a linear function of 3 combinations of the coefficients and variables, the total number of such combinations which appear in the 10 compounds being 13.

I take for the primitive the special form $(0, b, 2c, d, 0, 0, 0, 0, 1\between x, y)^8$, that is to say, $8bx^7y + 56cx^6y^2 + 56dx^5y^3 + y^8$, with the relation $bd = 3c^2$, and proceed to form the required derivatives in conformity with von Gall's scheme of derivation: I use, as the best practical method of obtaining the "alliance" of the i^{th} order between any two forms ϕ, ψ (of the orders μ, ν) denoted by $(\phi, \psi)_i$, the lineo-linear quadrinvariant (with respect to the variables of emanation) of the i^{th} emanant of ϕ combined into a system with the i^{th} emanant of ψ, taking care to reduce the result to the *parenthetical* form $(\ldots\ldots\between x, y)^{\mu+\nu-2i}$, containing only integer coefficients free from any common numerical factor. For the sake of brevity, too, I omit in general the symbolical factor containing (x, y): so that $(a_0, a_1, a_2, \ldots, a_i)$ will indicate the same thing as $(a_0, a_1, a_2, \ldots, a_i\between x, y)^i$. I shall adhere, in what follows, to the notation employed by Dr. von Gall.

We have then, according to this notation,

$$f = (0, b, 2c, d, 0, 0, 0, 0, 1) \quad \ldots \quad (1)$$

$$\begin{aligned} i = (f, f)_4 &= (4bx^3y + 12cx^2y^2 + 4dxy^3)\,y^4 - 4dx^4(bx^4 + 8cx^3y + 6dx^2y^2) \\ &\qquad + 3\,(2cx^4 + 4dx^3y)^2 \\ &= [12c^2 - 4bd,\ 16cd,\ 24d^2,\ 0,\ 0,\ 4b,\ 12c,\ 4d,\ 0][x, y]^8, \end{aligned}$$

where the square bracket is employed to signify the same thing as would be indicated by the use of the round clamp, with the exception that the binomial coefficients are suppressed. We have, therefore, introducing the multipliers $\frac{14}{1}, \frac{14}{8}, \frac{14}{28}, \frac{14}{56}, \frac{14}{70}, \frac{14}{56}, \frac{14}{28}, \frac{14}{8}, \frac{14}{1}$,

$$i = (0, 28cd, 12d^2, 0, 0, b, 6c, 7d, 0) \quad \ldots \quad (2)$$

$$\begin{aligned} k = (f, f)_6 &= (2bxy + 2cy^2)\,y^3 - 10d^2x^4 \\ &\equiv (\overline{20}d^2, 0, 0, b, 4c)^* \quad \ldots \quad (3) \end{aligned}$$

$$\begin{aligned} \Delta = (k, k)_2 &= \overline{20}d^2x^2(2bxy + 4cy^2) - b^2y^4 \\ &\equiv (0, 90c^2d, 40cd^2, 0, 3b^2) \quad \ldots \quad (4) \end{aligned}$$

$$C = (k, k)_4 = \overline{20}d^2 . 4c \equiv cd^2 \quad \ldots \quad (5)$$

$$\begin{aligned} f_4 = (f, k)_4 &= 4c(4bx^3y + 12cx^2y^2 + 4dxy^3) - 4b(bx^4 + 8cx^3y + 6d)x^2y^2 + \overline{20}d^2y^4 \\ &= -4b^2x^4 - 16bcx^3y + (48c^2 - 24bd)\,x^2y^2 + 16cdxy^3 - 20d^2y^4 \\ &\equiv (b^2, bc, c^2, \overline{cd}, 5d^2) \quad \ldots \quad (6) \end{aligned}$$

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* The sign of equivalence ($\equiv$) is used in the above and in what follows in the sense of "may be superseded by."

$$f_{k,2} = (f_4,\, k)_2 = (b^2x^2 + 2bcxy + c^2y^2)(2bxy + 4cy^2) - 2by^2(bcx^2 + 2c^2xy - cdy^2)$$
$$+ \overline{20}d^2x^2(c^2x^2 - 2cdxy + 5d^2y^2)$$
$$= [\overline{20}c^2d^2,\ 2b^3 + 40cd^3,\ 6b^2c + \overline{100}d^4,\ 6bc^2,\ 4c^3 + 2bcd][x,\ y]^4$$
$$\equiv (\overline{120}c^2d^2,\ 3b^3 + 60cd^3,\ 6b^2c + 100d^4,\ 9bc^2,\ 60c^3) \quad . \quad . \quad (7)$$
$$f_{k,3} = (f_4,\, k)_3 = (b^2x + bcy)(bx + 4cy) - 3by(bcx + c^2y) - \overline{20}d^2x(\overline{cd}x + 5d^2y)$$
$$= (b^3 + \overline{20}cd^3)x^2 + (2b^2c + 100d^4)xy + bc^2y^2$$
$$(f_{k,3})^2 = (b^6 + \overline{40}b^3cd^3 + 400c^2d^6)x^4 + (4b^5c\,[\overline{80}b^2c^2d^3 + 200b^3d^4] - 4000cd^7)x^3y$$
$$+ (6b^4c^2 + 400b^2cd^4 - 40bc^3d^3 + 10000d^8)x^2y^2$$
$$+ (4b^3c^3 + 200bc^2d^4)xy^3 + b^2c^4y^4$$
$$= [b^6 + \overline{1080}c^7 + \overline{400}c^2d^6,\ 4b^5c + 4680c^6d + \overline{4000}cd^7,$$
$$6b^4c^2 + 3480c^5d^3 + 10000d^8,\ 4b^3c^3 + 600c^4d^3,\ b^2c^4][x,\ y]^4$$
$$\equiv (3b^6 + \overline{3240}c^7 + 1200c^2d^6,\ 3b^5c + 3510c^6d + \overline{3000}cd^7,$$
$$3b^4c^2 + 1740c^5d^3 + 5000d^8,\ 3b^3c^3 + 450c^4d^3,\ 3b^2c^4) \ . \quad (8)$$
$$(f_\Delta) = (f,\, \Delta)_4 = 3b^2(4bx^3y + 12cx^2y^2 + 4dxy^3) + 6(40cd^2)(2cx^4 + 4dx^3y)$$
$$- 120bd^3(dx^4)$$
$$= [480c^2d^2 - 120bd^3,\ 12b^3 + 960cd^3,\ 36b^2c,\ 12b^2d,\ 0][x,\ y]^4$$
$$\equiv (40c^2d^2,\ b^3 + 80cd^3,\ 2b^2c,\ 3bc^2,\ 0) \ . \quad . \quad . \quad . \quad . \quad (9)$$
$$i_\Delta = (i,\, \Delta)_4 = -120bd^2(6bx^2y^2 + 24cxy^3 + 7dy^4) + 240cd^2(12d^2x^4 + 4bxy^3 + 6cy^4)$$
$$+ 3b^2(112cdx^3y + 72d^2x^2y^2)$$
$$= [2880cd^4,\ 336b^2cd,\ \overline{504}b^2d^2,\ \overline{1920}bcd^2,\ 1440c^2d^2 + \overline{840}bd^3][x,\ y]^4$$
$$\equiv (240cd^4,\ 21bc^3,\ \overline{63}c^4,\ \overline{120}c^3d,\ \overline{90}c^2d^2) \quad . \quad . \quad . \quad . \quad (10)$$
$$i_4 = (i,\, k)_4 = \overline{20}d^2(4bx^3y + 36cx^2y^2 + 28dxy^3) - 4\,.\,b(28cdx^4 + 48d^2x^3y + by^4)$$
$$+ (4c)(112cdx^3y + 72d^2x^2y^2)$$
$$= [\overline{112}bcd,\ 448c^2d + \overline{272}bd^2,\ \overline{432}cd^2,\ \overline{560}d^3,\ \overline{4b^2}][x,\ y]^4$$
$$\equiv (336c^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2) \quad . \quad . \quad . \quad . \quad . \quad (11)$$
$$i_{k,2} = (i_4,\, k)_2 = \overline{20}d^2x^2(72cd^2x^2 + 280d^3xy + 4b^2y^2)$$
$$- 2by^2(92c^2dx^2 + 144cd^2xy + 140d^3y^2)$$
$$+ (2bxy + 4cy^2)(336c^3x^2 + 184c^2dxy + 72cd^2y^2)$$
$$= [\overline{1440}cd^4,\ 672bc^3 + \overline{5600}d^5,\ 184bc^2d - 80b^2d^2 + 1344c^4,$$
$$736c^3d + \overline{144}bcd^2,\ 288c^2d^2 + \overline{280}bd^3][x,\ y]^4$$
$$\equiv (\overline{360}cd^4,\ 42bc^3 + \overline{350}d^5,\ 49c^4,\ 19c^3d,\ 138c^2d^2) \quad . \quad . \quad . \quad (12)$$
$$f_{k,\Delta} = (f_4,\, \Delta)_4 = -4(30bd^2)(\overline{cd}) + 6(40cd^2)c^2 + 3b^2\,.\,b^2$$
$$= 3b^4 + 120bcd^3 + 240c^3d^2$$
$$\equiv b^4 + 200c^3d^2 \ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$
$$i_{k,\Delta} = (i_4,\, \Delta)_4 = -4(30bd^2)(140d^3) + 6(40cd^2)(72cd^2) + (3b^2)(336c^3)$$
$$= 1008b^2c^3 + \overline{33120}c^2d^4 \equiv 7b^2c^3 - 230c^2d^4.$$

The term involving c^3d^4 being a multiple of the square of C (the invariant of the 4th degree) may be neglected, and, instead of $i_{k,\Delta}$, we may write the irreducible invariant of the 8th degree (say) $I_8 = b^2c^3$. . . . . (14)

That of the 7th degree we have just found $= b^4 + 200c^3d^3$; and obviously the quadrinvariant of f is identically zero, or say $I_2 = 0$. . . . (15)

Also the cubinvariant $I_3 = (f, i)_8$, where

$$f = (0, b, 2c, d, 0, 0, 0, 0, 1)$$

and $$i = (0, 28cd, 12d^2, 0, 0, b, 6c, 7d, 0).$$

Hence $$I_3 = -56bd + 336c^2 - 168bd = 504c^2 - 168bd = 0, \quad . \quad . \quad (16)$$

and we have found $I_4 \equiv cd^2$.

Also, $I_5 = (f, k^2)_8$ where

$$k^2 = (10d^2x^4 - 2bxy^3 - 2cy^4)^2$$

$$= 100d^4x^8 - 40bd^2x^5y^3 - 40cd^2x^4y^4 + 4b^2x^2y^6 + 8bcxy^7 + 4c^2y^8$$

Hence $$I_5 = 100d^4 + 2c.4b^2 - b.8bc \equiv d^4.*$$

The only remaining invariant required for present purposes is I_6, represented by $(i_4, k)_4$ where

$$k = [\overline{10}d^2, 0, 0, 2b, 2c][x, y]^4,$$

and

$$i_4 = (336c^3, 92c^2d, 72cd^2, 140d^3, 4b^2)(x, y)^4.$$

Hence

$$I_6 = \overline{40}b^2d^2 - (2b)92c^2d + 2c(336c^3)$$

$$= (-360 - 552 + 672)c^4 \equiv c^4.$$

On proceeding to form the 10 compound covariants of deg-order 10.4 obtained by suitable combinations of the invariants and covariants of inferior deg-order, it will be found that the following 13 arguments will make their appearance, in which, for greater brevity, x and y are each taken equal to unity, which in nowise affects (favorably or unfavorably) the course of the reasoning: these arguments are

$$b^6,\ c^7,\ c^2d^6;\quad b^5c,\ c^6d,\ cd^7;\quad b^4c^2,\ c^5d^2,\ d^8;\quad b^3c^3,\ c^4d^3;\quad b^2c^4,\ c^3d^4,$$

where the 5 groups of arguments, separated from one another by semicolons,

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* It will of course be recognized that the lineo-linear quadrinvariant to the system

$$(a_0, a_1, a_2, \ldots a_i)(x, y)^i, \quad [b_0, b_1, b_2, \ldots b_i][x, y]^i$$

is simply $$a_0b_i - a_1b_{i-1} + a_2 . b_{i-2} \ldots \pm a_ib_0:$$

the disappearance of the argument b^2c from companionship with d^4 in I_5 is rather remarkable, and could not have been predicted. This circumstance considerably simplifies the subsequent calculations.

are elements of the coefficients of x^4, x^3y, x^2y^2, xy^3, y^4, and when supplemented by such powers of k (of weight 8) as will bring their degrees up to the number 10, are of the respective weights 38, 39, 40, 41, 42, which is right, since the weight of the differentiant of deg-order 10 . 4 to $(x, y)^8$ is $\frac{10.8-4}{2}$, *i.e.* 38; for greater brevity (in what precedes) k, the coefficient of y^8 in f, has been made unity, and it is worthy of notice that all the arguments that can appear consistently with the law of weight are represented by these 13, upon the understanding that any power of bd in an argument is replaceable by the like power of c^2.

But it is further noticeable that the 10 compounds in question, although apparently linear functions of 13 arguments, are virtually such of only 11; for it will be seen that $b^6 + 4b^5c + 6b^4c^2$ may be regarded as a single argument, none of the three simpler arguments which appear in it occurring except in two of the 10 compounds, and their coefficients in each of those two being in the ratio $1:4:6$.

Had the contraction in the number of really independent arguments extended two steps further, so that the 10 compounds had been linear functions of only 9 quantities (as might, for anything that could be known *à priori*, have been the case), they would necessarily have been linearly connected, and no inference could have been drawn from the particular value assigned to f: moreover, had the 10 compounds been linear functions of only 10 quantities, although the particular form might have been sufficient for drawing a positive inference as to the non-existence of the general syzygy $\Omega = 0$, still there would have been no room for applying the all-important *test* of the correctness of the arithmetical computations upon which that inference would have reposed; and it would have been very unsatisfactory and unphilosophical to have made so important a conclusion rest upon the negative fact of a determinant of the 10th order *not vanishing*, when the undisproved existence of a single error committed in the many hundreds (or even—it might be said—thousands) of arithmetical steps involved in the calculations of the elements of that determinant would have been sufficient to account for its value differing from zero.

Fortunately, as will be seen, the correctness of the calculations may be *verified* (thanks to the existence of elements one more than barely sufficient—viz., 11 instead of 10) by the *positive* fact of a certain determinant of the 11th order being found equal to zero. It has often seemed to me that a special

providence or pre-established harmony in the intellectual world brings it about that honest labor, persevering in the pursuit of an important truth in the face of doubts and difficulties and repeated disappointments, shall not in the end lose its due reward.*

Let us now denote the quantities $b^6 + 4b^5c + 6b^4c^2$, c^7, c^2d^6; $4c^6d$, $4cd^7$; $6c^5d^2$, $6d^8$; $4b^3c^3$, $4c^4d^3$; b^2c^4, c^3d^4 by A, β, γ, δ, ε, ζ, η, θ, $\varkappa$, λ, μ, respectively, and denote the covariants of the order 4 that have been calculated in what precedes according to their deg-order—viz., let us call

$(f_{k,3})^2$; $i_{k,2}$; i_Δ; $f_{k,2}$; f_Δ; Δ; i_4; f_4; k
10 . 4; 6 . 4; 6 . 4*; 5 . 4; 5 . 4*; 4 . 4; 4 . 4*; 3 . 4; 2 . 4 respectively,

then the values of 10 . 4, $I_4 \times 6 . 4$, $I_4 \times 6 . 4^*$, $I_5 \times 5 . 4$, $I_5 \times 5 . 4^*$, $I_6 \times 4 . 4$, $I_6 \times 4 . 4^*$, $I_7 \times 3 . 4$, $I_4^2 \times 2 . 4$, $I_8 \times 2 . 4$, will be as shown in the table annexed

A	β	γ	δ	ε	ζ	η	θ	$\varkappa$	λ	μ	
3	$\overline{3240}$	1200	3510	$\overline{3000}$	1740	5000	3	450	3	.	. . (1)
.	.	$\overline{360}$	126	$\overline{350}$	49	.	.	19	.	$\overline{138}$	. . (2)
.	.	240	63	.	$\overline{63}$	.	.	$\overline{120}$	.	$\overline{90}$	. . (3)
.	.	$\overline{120}$	81	60	54	$\overline{100}$	.	27	.	60	. . (4)
.	.	40	27	80	18	.	.	9	.	.	. . (5)
.	.	.	90	.	40	.	.	.	3	.	. . (6)
.	336	.	92	.	72	.	.	140	4	.	. . (7)
1	1800	.	600	.	200	.	$\overline{3}$	$\overline{200}$	45	1000	. . (8)
.	.	$\overline{20}$	.	.	.	.	.	3	.	4	. . (9)
.	$\overline{180}$	.	.	.	.	.	1	.	4	.	. . (10)

Line (1) of course signifies $3A - 3240\beta + \ldots + 3\lambda$.
(2) . . . . $-360\gamma + 126\delta \ldots - 138\mu$,

and so for all the other lines, each being a linear function of the 11 quantities $A, \beta, \ldots, \lambda, \mu$.

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*I began with taking as a special form $ax^8 + by^8 + cz^8$, with the relation $x + y + z = 0$ (which, like the form f, contains two arbitrary ratios,) and went through the very considerable labor of calculating all its inferior derivatives capable of entering into the composition of a covariant of deg-order 10 . 4, but the result turned out altogether nugatory.

If these 10 linear functions are linearly connected, all the *complete* minors of the rectangular matrix (11 by 10) must vanish.

It is not so difficult as it might at first sight appear, to calculate the actual value of any one of these minors, convenient combinations of the lines and columns having been previously effected; this arises from the number of zeros which appear in the matrix. Mr. Morgan Jenkins, of the London Mathematical Society, and myself actually calculated two of them in the course of an hour or two; but the same object may be reached more expeditiously and quite as satisfactorily by proving that the minors do not vanish in respect to some judiciously or fortunately chosen modulus. I find that the number 11, taken as modulus, will accomplish the end in view. It will be found convenient to change the order of sequence of the lines and columns; to take the lines in the order 1, 8, 4, 10, 7, 6, 9, 5, 3, 2, and the columns in the order A, η, θ, β, λ, γ, δ, ε, ζ, $\varkappa$, μ. These transpositions having been effected, and the least positive residue of each element in respect to 11 being substituted in place of the element, the rectangular matrix above given will be replaced by the following:

3	6	3	5	3	1	1	3	2	10	.
1	.	8	7	1	.	6	.	2	9	10
.	10	.	.	.	1	4	5	10	5	5
.	.	1	7	4	.	.	.	.	.	.
.	.	.	6	4	.	4	.	6	8	.
.	.	.	.	3	.	2	.	7	.	.
.	.	.	.	.	2	.	.	.	3	4
.	.	.	.	.	7	5	3	7	9	.
.	.	.	.	.	9	8	.	8	1	9
.	.	.	.	.	3	5	2	5	8	5

It is easy to see that by proceeding as if to eliminate A between the two first lines, then β between the new line so formed and the third line, then γ between the new line again so formed and the fourth line, and so on, (always substituting the remainders to modulus 11 in lieu of the numbers themselves that arise in the process,) the first six lines may be replaced successively by the six following:

3	6	3	5	3	1	1	3	2	10	.
	5	10	5	.	10	6	8	4	6	8
		10	5	.	4	4	.	10	9	.
			10	7	7	7	.	1	2	.
				9	2	9	.	10	2	.
					5	2	.	.	5	.

Consequently, it only remains to ascertain whether the complete minors all disappear in the matrix of the dimensions (6×5) given below, viz:

$$\begin{matrix} 5 & 2 & . & . & 5 & . \\ 2 & . & . & . & 3 & 4 \\ 7 & 5 & 3 & 7 & 9 & . \\ 9 & 8 & . & 3 & 1 & 9 \\ 3 & 5 & 2 & 5 & 8 & 5 \end{matrix}$$

If all the complete minors of this matrix contain 11, the same must be true of the determinant formed by subtracting the first column in the above from the fifth and substituting the difference in place of the fifth column, *i. e.*

$$\begin{vmatrix} 2 & . & . & . & . \\ . & . & . & 1 & 4 \\ 5 & 3 & 7 & 2 & . \\ 8 & . & 3 & 3 & 9 \\ 5 & 2 & 5 & 5 & 5 \end{vmatrix} \quad \text{and therefore} \quad \begin{vmatrix} . & . & 1 & 4 \\ 3 & 7 & 2 & . \\ . & 3 & 3 & 9 \\ 2 & 5 & 5 & 5 \end{vmatrix}$$

should contain 11, and (as we may see by substituting the excess of 4 times the 3d column over the fourth in place of the 3d) the same must be true of the determinant $\begin{vmatrix} 3 & 7 & 8 \\ . & 3 & 3 \\ 2 & 5 & 4 \end{vmatrix}$ of which the value is $3(12-15)+2(21-24)$, *i. e.* -15, and as this does not contain 11, it follows that the complete minors of the matrix which expresses the 10 compounds as linear functions of the 11 arguments $\alpha, \beta, \gamma \ldots \lambda, \mu$ are not all zero, and they are consequently not linearly connected.* But, obviously, the calculations on which this proof

* In the *Comptes Rendus* for 22d August of this year, I have given a brief *résumé* of the contents of this paper. At page 367 of that fascicule, (third line from foot,) in the last line but one of the matrix, I have written 9 8 . 8 1 9 in error for 9 8 . 3 1 9 (having mistaken a 3, *covered with a blot*, for 8); consequently, the calculations which follow page 368 of the *C. R.* are erroneous. Fortunately, I did not repeat the mistake in calculating the value of the determinant subsequently given of the 11th order in proving that it contains the divisor 11. Moreover, this determinant, or rather its remainder to modulus 11, has been calculated by an entirely different process by Mr. Morgan Jenkins (whose work is before my eyes), and with the same result of its being divisible by 11. This instance shows how unsafe it would have been to have trusted to the fact of the minors not vanishing, unsupported by the positive evidence which the determinant of the 11th order affords of the preceding calculations, as regards the values of the groundforms, being unaffected with *one single error* in spite of the vast number of processes of addition, subtraction, multiplication, division, transposition, transcription and change of sign employed in working them out.

depends imperatively call for a verification, as nothing would be more easy than to bring out some or all of the minors different from zero by a single error of calculation or slip of the pen. To this end I calculate the value of von Gall's undecomposed covariant for the assumed special form f, and shall show that the 10 compounds and this 11th function do become linearly connected, *i. e.* subject to a syzygy, on the assumption that the arithmetical values of the coefficients have been correctly calculated.

The function in question, Dr. von Gall's i''_4, is obtained as follows:

$i'' = (i,\ \Delta)_2$ of deg-order 6.8 is equal to

$$
\begin{aligned}
&(168cdx^5y + 180d^2x^4y^2 + 6bxy^5 + 6cy^6)(40cd^2x^2 + 3b^2y^2)\\
&- (180c^2dx^2 + 160cd^2xy)(28cdx^6 + 72d^2x^5y + 15bx^2y^4 + 36cxy^5 + 7dy^6)\\
&+ (12d^2x^6 + 20bx^3y^3 + 90cx^2y^4 + 42dxy^5)(180c^2dxy + 40cd^2y^2)\\
&= [\overline{5040}c^3d^2,\ \overline{8560}c^2d^3,\ \overline{3840}cd^4,\ 504b^2cd,\ 7560c^4,\ 5640c^3d,\ 4380c^2d^2,\\
&\qquad\qquad 18b^3 + 560cd^3,\ 18b^2c]\,.\,[x,\ y]^8
\end{aligned}
$$

which, multiplied by 28, will be seen to be equivalent to

$$
\begin{aligned}
&(\overline{141120}c^3d^2,\ \overline{29960}c^2d^3,\ \overline{3840}cd^4,\ 756bc^3,\ 3024c^4,\ 2820c^3d,\ 4380c^2d^2,\\
&\qquad\qquad 63b^3 + 1960cd^3,\ 504b^2c).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
i''_4 = (i'',\ \Delta)_4 = {}& 3b^2\,(\overline{141120}c^3d^2x^4 + \overline{119840}c^2d^3x^3y + \overline{23040}cd^4x^2y^2\\
&\qquad + 3024bc^3xy^3 + 3024c^4y^4)\\
&+ 6\,.\,40cd^2\,(\overline{3840}cd^4x^4 + 3024bc^3x^3y + 18144c^4x^2y^2 + 11280c^3dxy^3 + 4380c^2d^2y^4)\\
&- 4\,.\,90c^2d(756bc^3x^4 + 12096c^4x^3y + 16920c^3dx^2y^2 + 17520c^2d^2xy^3 + (63b^3 + 1960cd^3)y^4
\end{aligned}
$$

which, dividing out by 144,

$$
\begin{aligned}
&\equiv (\overline{32130}c^7 + \overline{6400}c^2d^6)x^4 + \overline{37590}c^6dx^3y + \overline{16380}c^5d^2x^2y^2 + (63b^3c^3 + \overline{25000}c^4d^3)\,xy^3\\
&\qquad + \left(\overline{\frac{819}{2}}\,b^3c^4 + 2400c^3d^4\right)y^4\\
&\equiv (\overline{128520}c^7 + \overline{25600}c^2d^6,\ \ \overline{37590}c^6d,\ \ \overline{10920}c^5d^2,\ \ 63b^3c^3 + \overline{25000}c^4d^3,\\
&\qquad \overline{1638}b^3c^4 + 9600c^3d^4).
\end{aligned}
$$

Here it will be noticed that the arguments collected in what I have designated by A, viz. b^6, b^5c, b^4c^2, do not appear at all in i''_4. Had they made their appearance with other than coefficients bearing to each other the ratios of $1:4:6$, i''_4 could not have been a linear function of the 10 compounds which are linear functions of A and of 10 other arguments. This is in itself, to some extent, a verification of a portion at least of the preceding calculations: i''_4, as it turns

out, is a linear function of only 8 out of the 11 arguments which appear in the other 10 compound covariants, viz. of β, γ, δ, ζ, θ, $\varkappa$, λ, μ, neither A, ε nor η appearing in i_4''.

If the figuring throughout is correct, the determinant represented by the matrix constituted of the coefficients of the 11 compounds, ought to vanish identically; but it will be sufficient for all reasonable purposes (*i. e.* to satisfy any reasonable doubts on the subject) if I show that this is the case for the value of that determinant in respect to three consecutive prime numbers 11, 13, 17 taken almost at hazard.

It must be understood that the vanishing of the determinant in question adds *no additional strength whatever* to the proof—which, by Cayley's law, is perfect without it—provided that the figures in the coefficients of the 10 compounds (excluding i_4'') have been correctly calculated. It is to authenticate these figures, and not to verify the legitimacy of the argument, that the 11th compound is calculated, and the determinant formed by all the eleven shown to contain any number taken at will. It must be remembered that the calculations have been most carefully conducted and verified at each step: consequently, if any person, after the evidence that will be given, entertains any doubt of the correctness of the result, the duty is incumbent on him to put his finger upon some one of the coefficients of the 10 first compounds and prove it to be incorrectly stated.

First, for the modulus 11. In respect to this modulus, the coefficients in i_4'' of

$$A,\ \eta,\ \theta,\ \beta,\ \lambda,\ \gamma,\ \delta,\ \varepsilon,\ \zeta,\ \varkappa,\ \mu$$

are congruous to

$$0,\ 0,\ 8,\ 4,\ 1,\ 8,\ 8,\ 0,\ 3,\ 3,\ 8.$$

Hence, (making use of the transformations already calculated of the upper half of the rectangular matrix), it has to be shown that 11 is a divisor of the determinant of the 9th order

8	4	1	8	8	.	3	3	8
10	5	.	4	4	.	10	9	.
	10	7	7	7	.	1	2	.
		9	2	9	.	10	2	.
			5	2	.	.	5	.
			2	.	.	.	3	4
			7	5	3	7	9	.
			9	8	.	3	1	9
			3	5	2	5	8	5

The first and second lines of this matrix combined give rise to the line $1\ 7\ 7\ .\ 6\ 9\ 8$, and this, combined with the 4th, to the line $5\ 1\ .\ .\ 9\ 5$ under which last, writing the 5 remaining lines

$$\begin{array}{cccccc}
5 & 1 & . & . & 9 & 5 \\
5 & 2 & . & . & 5 & . \\
2 & . & . & . & 3 & 4 \\
7 & 5 & 3 & 7 & 9 & . \\
9 & 8 & . & 3 & 1 & 9 \\
3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

it has to be shown that the determinant to the above matrix of the 6th order contains 11.

Let the fourth line be replaced by 3 times itself + the last line, which, to the modulus 11, reduces the third column to the form of five zeros followed by 2. This shows that we may use, instead of the above, the determinant

$$\begin{array}{ccccc}
5 & 1 & . & 9 & 5 \\
5 & 2 & . & 5 & . \\
2 & . & . & 3 & 4 \\
2 & 9 & 4 & 2 & 5 \\
9 & 8 & 3 & 1 & 9
\end{array};$$

and again, replacing the fourth line of the new matrix by its double + the last line, we fall upon the matrix

$$\begin{array}{cccc}
5 & 1 & 9 & 5 \\
5 & 2 & 5 & . \\
2 & . & 3 & 4 \\
2 & 4 & 5 & 8
\end{array},$$

for which we may substitute

$$\begin{array}{cccc}
5 & 1 & 4 & 5 \\
5 & 2 & . & . \\
2 & . & 1 & 4 \\
2 & 4 & 3 & 8
\end{array},$$

or (as may be seen by replacing the second column by 3 times itself + the first column) $\begin{vmatrix} 8 & 4 & 5 \\ 2 & 1 & 4 \\ 3 & 3 & 8 \end{vmatrix}$, in which (to modulus 11) the first line is 4 times the second. Hence, the test is satisfied as regards the modulus 11.

I will next take the modulus 13.

The residues to modulus 13 of the coefficients in i_4'' of

$$\begin{array}{lccccccccc} & \theta & \beta & \lambda & \gamma & \delta & \varepsilon & \zeta & \varkappa & \mu \\ \text{will be seen to be} & 11 & 11 & . & 10 & 6 & . & . & 12 & 6. \end{array}$$

and the matrix corresponding to the one of the same dimensions (11×10), previously calculated for modulus 11, will, in respect to modulus 13, become

$$\begin{array}{ccccccccccc} 3 & 8 & 3 & 10 & 3 & 4 & . & 3 & 11 & 8 & . \\ 1 & . & 10 & 6 & 6 & . & 2 & . & 5 & 8 & 12 \\ & 4 & . & . & . & 10 & 3 & 8 & 2 & 1 & . \\ & & 1 & 2 & 4 & . & . & . & . & . & . \\ & & & 11 & 4 & . & 1 & . & 7 & 10 & . \\ & & & & 3 & . & 12 & . & 1 & 1 & . \\ & & & & & 6 & . & . & . & 3 & 4 \\ & & & & & 1 & 1 & 2 & 5 & 9 & . \\ & & & & & 6 & 11 & . & 2 & 10 & 1 \\ & & & & & 4 & 9 & 1 & 10 & 6 & 5. \end{array}$$

In place of the first six of the above lines, applying the same process as before, we may substitute

$$\begin{array}{ccccccccccc} 3 & 8 & 3 & 10 & 3 & 4 & . & 3 & 11 & 8 & . \\ & 5 & 1 & 8 & 2 & 9 & 5 & 10 & 4 & 3 & 10 \\ & & 9 & 7 & 5 & 1 & 8 & . & 7 & 6 & 12 \\ & & & 11 & 5 & 12 & 5 & 12 & 6 & 7 & 1 \\ & & & & 2 & 11 & 8 & 11 & 11 & 7 & 2 \\ & & & & & 6 & . & 7 & 8 & 7 & 7. \end{array}$$

Combining the i_4'' line (*i. e.* the coefficients of $\theta\ \beta\ \lambda \ldots \mu$ in i_4'' above given) with the third of these, we obtain the line

$$4 \quad 3 \quad 12 \quad 8 \quad . \quad 12 \quad 10 \quad .$$

which, again combined with the fourth of the same, gives rise to the line

$$7 \quad 10 \quad 9 \quad 9 \quad 9 \quad 4$$

Adding on the sixth line, viz. $6 \quad . \quad 7 \quad 8 \quad 7 \quad 7$ and the four last lines of the first matrix, viz. the lines marked with an asterisk,

$$\begin{array}{cccccc} *6 & . & . & . & 3 & 4 \\ *1 & 1 & 2 & 5 & 9 & . \\ *6 & 11 & . & 2 & 10 & 1 \\ *4 & 9 & 1 & 10 & 6 & 5, \end{array}$$

the arithmetical problem to be solved reduces itself to showing that the above determinant vanishes to modulus 13.

Substituting for the 1st column twice the 1st less three times the 6th, and for the 5th column twice the 5th less the 1st, and neglecting the factor 3, we fall upon the determinant

$$\begin{vmatrix} 2 & 10 & 9 & 9 & 11 \\ 4 & . & 7 & 8 & 8 \\ 2 & 1 & 2 & 5 & 4 \\ 9 & 11 & . & 2 & 1 \\ 6 & 9 & 1 & 10 & 8 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 2 & 10 & 9 & 9 & 2 \\ 4 & . & 7 & 8 & . \\ 2 & 1 & 2 & 5 & 12 \\ 9 & 11 & . & 2 & 12 \\ 6 & 9 & 1 & 10 & 11 \end{vmatrix}.$$

Then in this last, substituting for the 4th column the 4th less twice the 1st, say M, and for the 3d column 5 times the 1st less the 3d, say N, we descend in like manner upon the determinant

$$\begin{array}{cccc} 5 & 2 & 2 & 1 \\ 1 & 2 & 12 & 8 \\ 10 & 9 & 12 & 6 \\ 11 & 6 & 11 & 3 \end{array}$$

where the 1st column is the M with the zero in it left out, and the 4th column the N with the zero in it left out.

This, by elimination (so to say) of the first variable to the left between the successive pairs of lines, gives rise to the determinant

$$\begin{array}{ccc} 8 & 6 & . \\ 2 & 9 & 4 \\ . & 4 & 3 \end{array}$$

which (to modulus 13) $\equiv 8 \,.\, 1 - 8 \,.\, 3 - 6 \,.\, 6 \equiv 8 - 11 - 10 \equiv 0$.

It remains only to apply the 3d proposed test, using 17 as the modulus.

The i_4'' line here becomes

$$\begin{array}{ccccccccc} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \end{array}$$

and the grand rectangular matrix becomes

$$\begin{array}{ccccccccccc} 3 & 2 & 3 & 7 & 3 & 10 & 8 & 9 & 6 & 8 & . \\ 1 & . & 14 & 15 & 11 & . & 5 & . & 13 & 4 & 14 \\ & 2 & . & . & . & 16 & 13 & 9 & 3 & 10 & 9 \\ & & 1 & 7 & 4 & . & . & . & . & . & . \\ & & & 13 & 4 & . & 7 & . & 4 & 4 & . \\ & & & & 3 & . & 5 & . & 6 & . & . \end{array}$$

with 4 more lines, which will be presently supplied in their proper place. For those above written may be substituted

3	2	3	7	3	10	8	9	6	8	.
	15	5	4	13	7	7	8	16	4	8
		7	9	8	5	11	.	13	6	.
			6	3	12	6	.	4	11	.
				2	14	15	.	6	.	.
					9	16	.	11	.	.

Rejecting the first two lines, and writing over the remaining ones the i_4'' line, there results

12	0	11	2	14	.	11	7	12
7	9	8	5	11	.	13	6	.
	6	3	12	6	.	4	11	.
		2	14	15	.	6	.	.
			9	16	.	11	.	.

which may be replaced by

12	0	11	2	14	.	11	7	12
	6	2	12	.	.	11	6	1
		6	.	2	.	9	13	11
			16	1	.	1	8	12
			9	16	.	11	.	.

to which I add in the 4 pretermitted lines distinguished by asterisks,

*14	.	.	.	3	4
* 6	10	12	1	9	.
* 2	12	.	5	16	12
*14	7	7	15	2	15

and the determinant, represented by the square matrix (6×6) exhibited by the 6 lines last appearing above, ought to contain the modulus 17 as a divisor. Instead of the 3d line from the bottom we may substitute its double less the last line, and thus, neglecting the factor 7, fall upon the matrix

16	1	1	8	12
9	16	11	.	.
14	.	.	3	4
15	13	4	16	2
2	12	5	16	12.

Substituting for the 4th column the sum of itself and the 1st, and for the 5th column 5 times itself + the 1st, and neglecting the factor 14, we obtain the determinant

$$\begin{matrix} 1 & 1 & 7 & 8 \\ 16 & 11 & 9 & 9 \\ 13 & 4 & 14 & 8 \\ 12 & 5 & 1 & 11 . \end{matrix}$$

Subtracting the 2d column from the 1st and the 4th from the 2d + the 3d, we obtain the matrix

$$\begin{matrix} 0 & 0 & 1 & 7 \\ 5 & 11 & 11 & 9 \\ 9 & 10 & 4 & 14 \\ 7 & 12 & 5 & 1, \end{matrix}$$

and replacing the 3d column by 7 times the 3d less the 4th, we descend upon the determinant

$$\begin{matrix} 5 & 11 & . \\ 9 & 10 & 14 \\ 7 & 12 & . \end{matrix}$$ where the 1st line to modulus 17 equals 8 times the 3d.

Hence the determinant in question contains 17, as was to be shown.

It seems needless to multiply these tests—the object being, as before stated, not a confirmation of the argument, which is wholly unnecessary, but a verification of the accuracy of the arithmetic: for this reason it has seemed to me essential that the calculations, authenticating the figures previously obtained, should be set out in considerable detail.

Instead of founding anything upon the concordance (as far as it extends) between Dr. von Gall's table and my own, the proof of the non-existence of the 10.4 irreducible covariant may be inferred exclusively from the former and completed as follows.

I have proved that the syzygetic function Ω of the deg-order 10.4, if it exists, must be a consequence of the existence of a like function of the deg-order 8.4, 7.4, or 5.4. The last hypothesis may at once be rejected as implying an equation of the form $\frac{2.4}{3.4}$ = a numerical multiple of $\frac{2.0}{3.0}$.

Next, for the deg-order 7.4, again using for the primitive the same special form f, which causes 2.0 and 3.0 to vanish, the only non-vanishing argu-

ments in the supposed syzygetic function Ω' for the particular form f will be 4.0×3.4 and 5.0×2.4, *i. e.* $cd^3(b^3, bc, c^2, -cd, 5d^2)$, and $d^4(20d^2, 0, 0, b, 4c)$, between which obviously no syzygy is possible, so that neither of them can appear in the general form of Ω'. Hence the terms in the general form of Ω' must be divisible all by 2.0 or all by 3.0, or some by 2.0 and some by 3.0, and consequently there must exist a syzygy of the deg-order 5.4, 4.4, or 2.4. The first of these hypotheses has already been shown to be impossible, and the remaining two need not even have been mentioned, as there is only a single compound of the deg-order 4.4, viz. 2.0×2.4, and none of the deg-order 2.4. Lastly, for the deg-order 8.4, still using the same special form of f, the arguments in the supposed syzygetic functions which do not vanish are 4.0×4.4, $4.0 \times 4.4^*$, 5.0×3.4, and 6.0×2.4, *i. e.*

$$cd^2(0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^3)$$
$$cd^3(336c^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^3)$$
$$d^4(b^3,\ bc,\ c^2,\ -cd,\ 5d^2)$$

and

$$c^4(-20d^2,\ 0,\ 0,\ b,\ 4c).$$

The argument d^6 in the 3d of these quantities has no equivalent in any of the other 3. Hence the 3d quantity does not appear in the syzygy: moreover, the 4th compound contains one argument, viz. bc^4, which does not rationally contain $d^2c\left(\text{for } \frac{bc^3}{d^2} = \frac{b^2c}{3d}\right)$. Hence this compound also disappears, and obviously no syzygy connects together the first two. Hence in the supposed general syzygy there exist no compounds containing neither 2.0 nor 3.0, and by the same reasoning as before, this supposed syzygetic function must imply the existence of one of the deg-order 6.4 or 5.4 or 4.4. The two last of the three suppositions have already been seen to be impossible, and the first would imply a linear relation between 2.0×4.4, $2.0\times 4.4^*$, 3.0×3.4, 4.0×2.4, the last of which we see, by taking f for the primitive, cannot appear in the general syzygy, and the remaining 3 arguments would imply that the general covariant 3.4 would contain the invariant 2.0, which is absurd. Hence it follows from Dr. von Gall's own results that the existence of a groundform of deg-order 10.4 is impossible. The only principle extraneous to his results made use of is Cayley's all-important rule, of which an irrefragable demonstration has been given by the author of this paper, but which still, as far as he is aware, remains unutilized, and is almost passed over in silence by invariantists of the German school.

It may be as well to make this article self-contained by showing that the number of compound irreducible groundforms of deg-order 10 . 4 is, as stated, 32, viz. the same as the number of linearly-independent covariants of that deg-order requisitioned by Cayley's rule.

Using then, for brevity's sake, i to represent the invariant $i\,.\,0$, it is easy to see that the following is an exhaustive enumeration of all the compounded irreducibles of deg-order 10 . 4:

$(5.2)^2$; 8×2.4; 7×3.4; 6×4.4; $6\times4.4^*$; 5×5.4; $5\times5.4^*$; 4×6.4; $4\times6.4^*$; $4\times4\times2.4$; 3×7.4; $3\times7.4^*$; $3^2\times4.4$; $3^2\times4.4^*$; $3\times4\times3.4$; $3\times5\times2.4$; 2×8.4; $2\times8.4^*$; $2\times3\times5.4$; $2\times3\times5.4^*$; $2\times4\times4.4$; $2\times4\times4.4^*$; $2\times5\times3.4$; $2\times6\times2.4$; $2\times3^2\times2.4$; $2^2\times6.4$; $2^2\times6.4^*$; $2^2\times3\times3.4$; $2^2\times4\times2.4$; $2^3\times4.4$; $2^3\times4.4^*$; $2^4\times2.4$.

The same number 32, it is all-important to bear in mind, is also the number of linearly independent covariants of deg-order 10 . 4 given by Cayley's law. For this number is represented by $(w:8,10)-(w':8,10)$ where $w=\dfrac{10.8-4}{2}=38$, $w'=w-1=37$; *i. e.* (by Euler's Theorem) is the coefficient of t^{38} in the development of

$$\frac{(1-t^{11})(1-t^{12})(1-t^{13})(1-t^{14})(1-t^{15})(1-t^{16})(1-t^{17})(1-t^{18})}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)(1-t^8)},$$

which may be calculated as follows: The numerator is $1-t^{11}-t^{12}-t^{13}-t^{14}-t^{15}-t^{16}-t^{17}-t^{18}+t^{23}+t^{24}+2t^{25}+2t^{26}+3t^{27}+3t^{28}+4t^{29}+3t^{30}+3t^{31}+2t^{32}+2t^{33}+t^{34}+t^{35}-t^{36}-t^{37}-2t^{38}\ldots\ldots$ Dividing this by $1-t^8$, the quotient by $1-t^7$, and so on for $1-t^6, \ldots . 1-t^2$, we have for the numerator and the successive quotients so obtained the following values respectively:

t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}	t^{14}	t^{15}	t^{16}	t^{17}	t^{18}	t^{19}	t^{20}	t^{21}	t^{22}	t^{23}	t^{24}	t^{25}	t^{26}	t^{27}	t^{28}	t^{29}	t^{30}	t^{31}	t^{32}	t^{33}	t^{34}	t^{35}	t^{36}	t^{37}	t^{38}
1	0	0	0	0	0	0	0	0	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	0	1	1	2	2	3	3	4	3	3	2	2	1	1	$\bar{1}$	$\bar{1}$	$\bar{2}$
1	0	0	0	0	0	0	0	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	1	1	1	2	2	3	2	3	3	3	2	3	1	2	0
1	0	0	0	0	0	0	1	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	0	0	$\bar{1}$	$\bar{1}$	0	1	2	2	3	2	2	2	4	3	4	3
1	0	0	0	0	0	1	1	1	0	0	$\bar{1}$	0	0	1	0	0	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	0	0	0	0	0	1	2	4	3	4	3
1	0	0	0	0	1	1	1	1	0	1	0	1	1	1	1	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	1	0	1	1
1	0	0	0	1	1	1	1	2	1	2	1	3	2	3	2	3	1	2	1	3	0	0	$\bar{2}$	0	$\bar{3}$	$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{6}$	$\bar{6}$	$\bar{8}$	$\bar{6}$	$\bar{8}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{7}$	$\bar{6}$
1	0	0	1	1	1	2	2	3	3	4	4	6	6	7	8	9	8	10	10	11	10	10	9	10	7	6	5	4	0	$\bar{1}$	$\bar{4}$	$\bar{6}$	$\bar{9}$	$\overline{11}$	$\overline{13}$	$\overline{15}$	$\overline{18}$	$\overline{19}$
1	0	1	1	2	2	4	4	7	7	11	11	17	17	24	25	33	33	43	43	54	53	64	62	74	69	80	74	84	74	83	70	77	61	66	48	51	30	32

Hence the required coefficient is 32.

It is obvious that the particular method adopted in treating the grand determinant made up of 11^2 places employed in the foregoing investigation furnishes or indicates a good practical process for determining 10 out of the 32 numerical coefficients which enter into the expression of Dr. von Gall's covariant i_4'' as a linear function of the 32 linearly independent covariants of its own deg-order; but, as this calculation possesses no point either of intrinsic theoretical interest or practical importance, I leave it to those who may feel any curiosity on the subject, to go through the calculations necessary to attain that end.

It may be supposed that the long calculations rendered necessary by the quadrinomial form f, attributed to the primitive in the preceding investigation, might have been evaded by using a trinomial form (of which several exist) possessing the same property of causing the two first invariants to vanish, and not less general, inasmuch as containing three independent coefficients in place of four connected by a homogeneous equation; *e. g.* we might assume for the primitive $(0, b, 0, 0, 0, f, 0, 0, i \between x, y)^8$, where the weights of b, f, i are respectively $1, 5, 8$.

The quadrinvariant vanishes because no binary combination of $1, 5, 8$, with or without repetitions, will make up the required weight 8, and the cubinvariant because no ternary combination of the same will make up the weight 12. It may, however, easily be shown that such form will lead only to a nugatory conclusion, as not supplying the necessary number of arguments (10 at least are wanted) to support the independence of the 10 surviving compound covariants of deg-order 10.4. This may be seen as follows.

The weights of the coefficients of x^4, x^3y, x^2y^2, xy^3, y^4 in a 10.4 covariant are respectively 38, 39, 40, 41, 42. Let us ascertain in how many ways 10 numbers, consisting exclusively of the numbers $1, 5, 8$, can be put together to make up these totals. I use the notation $a^\alpha . b^\beta . c^\gamma$ to indicate a sum of α numbers a, β numbers b, and γ numbers c.

Then the sole admissible representations of 38 are $8^4 . 1^6$, $5^7 . 1^3$,

" 39 " $8^3 . 5^2 . 1^5$,

" 40 " $8^2 . 5^4 . 1^4$,

" 41 " $8 . 5^6 . 1^3$,

" 42 " $8^4 . 5 . 1^5$, $5^8 . 1^2$,

i. e. there are only at utmost 7 arguments contained in the expressions for the 10 compounds.

So, in like manner, if we assumed for the primitive

$$(0, b, 0, 0, 0, 0, g, 0, i \between x, y)^8$$

to find the number of independent arguments possible in a 10 . 4 covariant, we must ascertain the sum of the numbers of similar representations to the foregoing of the same integers 38, 39, 40, 41, 42, with 10 integers confined to be 1, 6 or 8, and we shall find that the sole representations of that kind are $8^4 . 1^6$; $8^2 . 6^3 . 1^5$; $6^6 . 1^4$; $8^4 . 6 . 1^5$; $8^3 . 6^2 . 1^5$; $8 . 6^5 . 1^4$, *i. e.* 6 representations in all. In like manner it will be found that all the other trinomial forms of the primitive so taken that the first two invariants are null, will be incapable of yielding as many as 10 arguments to any covariant of deg-order 10 . 4,* so that the 10 compounds appurtenant to such special form will be bound to be linearly related, and no inference can be drawn from any such assumption. I have reason for believing that the quadrinomial form employed

---

*On an exhaustive examination, it will be found that the only trinomial forms of the primitive which will cause the first two invariants to disappear, are those in which the surviving coefficients are

b, f, i; b, g, i
a, b, c; a, b, d; a, c, d; b, c, d,

or the complementary ones

g, d, a; h, c, a
i, h, g; i, h, f; i, g, f; h, g, f,

which, of course, are substantially equivalent to the former.

Confining our attention, then, to the upper group, it will readily be seen that the four last will cause not only the quadrinvariant and the cubinvariant, but all the other invariants as well, to vanish. Since, then, it has been shown that the b, f, i; b, g, i forms are insufficient to support the independence of the 10 compound covariants with which the reasoning is concerned, it follows that *no trinomial form* will be adequate to do so.

It may be asked what would have been the effect of using the form in which b, c, d, i are the surviving coefficients, but b, c, d are supposed mutually independent, instead of being subject to the condition employed in the refutation above: on this supposition the quadrinvariant, but not the cubinvariant, will vanish; and an easy calculation will show that of the 32 representations of the covariant of deg-order 10 . 4 as a product of inferior groundforms there will be only 16 in which the quadrinvariant does not appear as a factor. And, again, it will be found that the number of ways of representing 38, 39, 40, 41, 42, as the sum of 4 numbers, each of which is either 1, 2, 3 or 8 is 29. Hence there would arise a matrix of 16 lines and 29 columns, and to disprove the existence of the 10 . 4 groundform it would be sufficient to prove that some one of the *complete* minor determinants of this matrix differs from zero. The work involved in dealing with this and the subsequent verificatory matrix of 17 lines and 29 columns would evidently be vastly greater and more liable to error than when (as in the text) we assign the relation between b, c, d so as to make the cubinvariant vanish.

In the absence of the information as to the number of linearly independent 10 . 4's given by Cayley's rule, the direct mode of refutation would have required the calculation of the 32 compound 10 . 4's and the problematical one of von Gall for the general form of the Octavic, subject only to the simplification of taking two of the coefficients zero. There would then have remained to show that the leading terms of these 33 forms were linearly connected, which would necessarily imply that the same was true of the 33 entire forms themselves; a colossal task, probably transcending the sphere of human ability to execute.

in the foregoing investigation is the most convenient and economical, as leading to the simplest calculations of any that could have been employed for the same purpose.

It may be well (by way of confirmation) to determine *à priori* the number of possible arguments that can belong to the 10 . 4 covariants of the quadrinomial form of $(x, y)^8$ employed in the antecedent investigation. Since c^2 may be replaced by a numerical multiple of bd, it follows that each argument may be brought to a form in which c does not enter at all, or in which it enters only in the first degree. The total possible number (which turns out to be the actual number) of arguments is, consequently, the number of ways in which 38, 39, 40, 41, 42 can be composed with 10 parts each of them 1, 3 or 8 + the number of ways in which 36, 37, 38, 39, 40 can be composed with 9 parts, each of them also 1, 3 or 8. All the possible different compositions of these kinds are exhibited in the annexed table.

$38 = 4.8 + 6.1 = 2.8 + 7.3 + 1.1$	$36 = 3.8 + 3.3 + 3.1$
$39 = 3.8 + 4.3 + 3.1$	$37 = 4.8 + 5.1 = 2.8 + 7.3$
$40 = 4.8 + 5.1 + 1.3 = 2.8 + 8.3$	$38 = 3.8 + 4.3 + 2.1$
$41 = 3.8 + 5.3 + 2.1$	$39 = 4.8 + 1.3 + 4.1$
$42 = 4.8 + 4.1 + 2.3$	$40 = 3.8 + 5.3 + 1.1.$

There are thus 7 + 6 *i. e.* 13 distinct arguments, *i. e.* the number which actually appear distributed among the 10 surviving covariants of deg-order 10 . 4 as previously shown—it being at the same time remembered that three of the 13 enter as elements of a fixed linear combination into the 10 functions, which are thus virtually functions of only 11 independent arguments.

The method employed in what precedes suggests a mode of calculating in part at least the discriminant of the eighthic in terms of the subordinate groundforms. Thus, suppose we take for our special form, $(0, b, c, d, 0, 0, 0, 0, i \between x, y)^8$ with b, c, d independent.

Then the quadrinvariant will vanish, and there will be no very great effort of calculation required to express the 8 remaining invariants as functions of b, c, d, i.

The discriminant is of the 14th degree and 14 may be made up in 10 (and no more than 10) ways as a sum of numbers each limited to be 3, 4, 5, 6, 7, 8, 9, or 10; as exhibited in the exhaustive table

$$14 = 10 + 4 = 9 + 5 = 8 + 6 = 8 + 3 + 3 = 7 + 7 = 7 + 4 + 3 = 6 + 4 + 4$$
$$= 6 + 5 + 3 = 5 + 5 + 4 = 4 + 4 + 3 + 3.$$

Again the weight of the discriminant is 56, and the number of ways of compounding 56 with 14 numbers each limited to be 1, 2, 3 or 8 is 11, as shown in the exhaustive table

$$56 = 6.8 + 8.1 = 5.8 + 7.2 + 2.1 = 5.8 + 3.3 + 1.2 + 5.1 = 5.8 + 2.3 + 3.2 + 4.1 = 5.8 + 1.3 + 5.2 + 3.1 = 5.8 + 7.2 + 2.1 = 4.8 + 7.3 + 3.1 = 4.8 + 6.3 + 2.2 + 2.1 = 4.8 + 5.3 + 4.2 + 1.1 = 4.8 + 4.3 + 6.2 = 3.8 + 10.3 + 1.2$$

Now there will be no difficulty at all in finding by substitution and multiplication the discriminant of the assumed quantic, say Q, which is in fact the same as the resultant of $\frac{dQ}{dy}$ and $bx^6y + 3cx^5y^2 + 5dx^4y^3$. Hence there will be 11 equations for determining the coefficients of the 10 invariants of the 14th degree which are products of the inferior invariants (the quadrinvariant excepted); consequently there will be sufficient or more than sufficient equations for the purpose, unless it should (unfortunately and contrary to probability) turn out to be the case that the 10 products, although linear functions of 11 arguments, are expressible as linear functions of only 9 linear functions of those arguments.

On the Logic of Number

By C. S. Peirce.

Nobody can doubt the elementary propositions concerning number: those that are not at first sight manifestly true are rendered so by the usual demonstrations. But although we see they *are* true, we do not so easily see precisely *why* they are true; so that a renowned English logician has entertained a doubt as to whether they were true in all parts of the universe. The object of this paper is to show that they are strictly syllogistic consequences from a few primary propositions. The question of the logical origin of these latter, which I here regard as definitions, would require a separate discussion. In my proofs I am obliged to make use of the logic of relatives, in which the forms of inference are not, in a narrow sense, reducible to ordinary syllogism. They are, however, of that same nature, being merely syllogisms in which the objects spoken of are pairs or triplets. Their validity depends upon no conditions other than those of the validity of simple syllogism, unless it be that they suppose the existence of singulars, while syllogism does not.

The selection of propositions which I have proved will, I trust, be sufficient to show that all others might be proved with like methods.

Let r be any relative term, so that one thing may be said to be r of another, and the latter r'd by the former. If in a certain system of objects, whatever is r of an r of anything is itself r of that thing, then r is said to be a transitive relative in that system. (Such relatives as "lover of everything loved by —" are transitive relatives.) In a system in which r is transitive, let the q's of anything include that thing itself, and also every r of it which is not r'd by it. Then q may be called a fundamental relative of quantity; its properties being, first, that it is transitive; second, that everything in the system is q of itself, and, third, that nothing is both q of and q'd by anything except itself. The objects of a system having a fundamental relation of quantity are called quantities, and the system is called a system of quantity.

A system in which quantities may be q's of or q'd by the same quantity without being either q's of or q'd by each other is called multiple;* a system in which of every two quantities one is a q of the other is termed simple.

Simple Quantity.

In a simple system every quantity is either "as great as" or "as small as" every other; whatever is as great as something as great as a third is itself as great as that third, and no quantity is at once as great as and as small as anything except itself.

A system of simple quantity is either continuous, discrete, or mixed. A continuous system is one in which every quantity greater than another is also greater than some intermediate quantity greater than that other. A discrete system is one in which every quantity greater than another is next greater than some quantity (that is, greater than without being greater than something greater than). A mixed system is one in which some quantities greater than others are next greater than some quantities, while some are continuously greater than some quantities.

Discrete Quantity.

A simple system of discrete quantity is either limited, semi-limited, or unlimited. A limited system is one which has an absolute maximum and an absolute minimum quantity; a semi-limited system has one (generally considered a minimum) without the other; an unlimited has neither.

A simple, discrete, system, unlimited in the direction of increase or decrement, is in that direction either infinite or super-infinite. An infinite system is one in which any quantity greater than x can be reached from x by successive steps to the next greater (or less) quantity than the one already arrived at. In other words, an infinite, discrete, simple, system is one in which, if the quantity next greater than an attained quantity is itself attained, then any quantity greater than an attained quantity is attained; and by the class of attained quantities is meant any class whatever which satisfies these conditions. So that we may say that an infinite class is one in which if it is true that every quantity next greater than a quantity of a given class itself belongs to that class, then it is true that every

* For example, in the ordinary algebra of imaginaries two quantities may both result from the addition of quantities of the form $a^2 + b^2i$ to the same quantity without either being in this relation to the other.

quantity greater than a quantity of that class belongs to that class. Let the class of numbers in question be the numbers of which a certain proposition holds true. Then, an infinite system may be defined as one in which from the fact that a certain proposition, if true of any number, is true of the next greater, it may be inferred that that proposition if true of any number is true of every greater.

Of a super-infinite system this proposition, in its numerous forms, is untrue.

Semi-infinite Quantity.

We now proceed to study the fundamental propositions of semi-infinite, discrete, and simple quantity, which is ordinary number.

Definitions.

The minimum number is called one.

By $x + y$ is meant, in case $x = 1$, the number next greater than y; and in other cases, the number next greater than $x' + y$, where x' is the number next smaller than x.

By $x \times y$ is meant, in case $x = 1$, the number y, and in other cases $y + x'y$, where x' is the number next smaller than x.

It may be remarked that the symbols $+$ and $\times$ are triple relatives, their two correlates being placed one before and the other after the symbols themselves.

Theorems.

The proof in each case will consist in showing, 1st, that the proposition is true of the number one, and 2d, that if true of the number n it is true of the number $1 + n$, next larger than n. The different transformations of each expression will be ranged under one another in one column, with the indications of the principles of transformation in another column.

1. To prove the associative principle of addition, that

$$(x + y) + z = x + (y + z)$$

whatever numbers x, y, and z, may be. First it is true for $x = 1$; for

$(1 + y) + z$

$= 1 + (y + z)$ by the definition of addition, 2d clause. Second, if true for $x = n$, it is true for $x = 1 + n$; that is, if $(n + y) + z = n + (y + z)$ then $((1 + n) + y) + z = (1 + n) + (y + z)$. For

$$
\begin{aligned}
&((1+n)+y)+z \\
&=(1+(n+y))+z \quad \text{by the definition of addition:} \\
&=1+((n+y)+z) \quad \text{by the definition of addition:} \\
&=1+(n+(y+z)) \quad \text{by hypothesis:} \\
&=(1+n)+(y+z) \quad \text{by the definition of addition.}
\end{aligned}
$$

2. To prove the commutative principle of addition that

$$x+y=y+x$$

whatever numbers x and y may be. First, it is true for $x=1$ and $y=1$, being in that case an explicit identity. Second, if true for $x=n$ and $y=1$, it is true for $x=1+n$ and $y=1$. That is, if $n+1=1+n$, then $(1+n)+1=1+(1+n)$. For $(1+n)+1$

$$
\begin{aligned}
&=1+(n+1) \quad \text{by the associative principle:} \\
&=1+(1+n) \quad \text{by hypothesis.}
\end{aligned}
$$

We have thus proved that, whatever number x may be, $x+1=1+x$, or that $x+y=y+x$ for $y=1$. It is now to be shown that if this be true for $y=n$, it is true for $y=1+n$; that is, that if $x+n=n+x$, then $x+(1+n)=(1+n)+x$. Now,

$$
\begin{aligned}
&x+(1+n) \\
&=(x+1)+n \quad \text{by the associative principle:} \\
&=(1+x)+n \quad \text{as just seen:} \\
&=1+(x+n) \quad \text{by the definition of addition:} \\
&=1+(n+x) \quad \text{by hypothesis:} \\
&=(1+n)+x \quad \text{by the definition of addition.}
\end{aligned}
$$

Thus the proof is complete.

3. To prove the distributive principle, first clause. The distributive principle consists of two propositions:

$$
\begin{aligned}
&\text{1st,} \quad (x+y)\,z=xz+yz \\
&\text{2d,} \quad x\,(y+z)=xy+xz.
\end{aligned}
$$

We now undertake to prove the first of these. First, it is true for $x=1$. For

$$
\begin{aligned}
&(1+y)\,z \\
&=z+yz \quad \text{by the definition of multiplication:} \\
&=1\,.\,z+yz \quad \text{by the definition of multiplication.}
\end{aligned}
$$

Second, if true for $x = n$, it is true for $x = 1 + n$; that is, if $(n + y)z = nz + yz$, then $((1 + n) + y)z = (1 + n)z + yz$. For

$$
\begin{array}{ll}
((1 + n) + y)z & \\
= (1 + (n + y))z & \text{by the definition of addition:} \\
= z + (n + y)z & \text{by the definition of multiplication:} \\
= z + (nz + yz) & \text{by hypothesis:} \\
= (z + nz) + yz & \text{by the associative principle of addition:} \\
= (1 + n)z + yz & \text{by the definition of multiplication.}
\end{array}
$$

4. To prove the second proposition of the distributive principle, that

$$x(y + z) = xy + xz.$$

First, it is true for $x = 1$; for

$$
\begin{array}{ll}
1(y + z) & \\
= y + z & \text{by the definition of multiplication:} \\
= 1y + 1z & \text{by the definition of multiplication.}
\end{array}
$$

Second, if true for $x = n$, it is true for $x = 1 + n$; that is, if $n(y + z) = ny + nz$, then $(1 + n)(y + z) = (1 + n)y + (1 + n)z$. For

$$
\begin{array}{ll}
(1 + n)(y + z) & \\
= (y + z) + n(y + z) & \text{by the definition of multiplication:} \\
= (y + z) + (ny + nz) & \text{by hypothesis:} \\
= (y + ny) + (z + nz) & \text{by the principles of addition:} \\
= (1 + n)y + (1 + n)z & \text{by the definition of multiplication.}
\end{array}
$$

5. To prove the associative principle of multiplication; that is, that

$$(xy)z = x(yz),$$

whatever numbers x, y, and z, may be. First, it is true for $x = 1$, for

$$
\begin{array}{ll}
(1y)z & \\
= yz & \text{by the definition of multiplication:} \\
= 1.yz & \text{by the definition of multiplication.}
\end{array}
$$

Second, if true for $x = n$, it is true for $x = 1 + n$; that is, if $(ny)z = n(yz)$, then $((1 + n)y)z = (1 + n)(yz)$. For

$$
\begin{aligned}
&((1 + n)\, y)\, z \\
&= (y + ny)\, z && \text{by the definition of multiplication:} \\
&= yz + (ny)\, z && \text{by the distributive principle:} \\
&= yz + n\,(yz) && \text{by hypothesis:} \\
&= (1 + n)\,(yz) && \text{by the definition of multiplication.}
\end{aligned}
$$

6. To prove the commutative principle of multiplication; that

$$xy = yx,$$

whatever numbers x and y may be. In the first place, we prove that it is true for $y = 1$. For this purpose, we first show that it is true for $y = 1$, $x = 1$; and then that if true for $y = 1$, $x = n$, it is true for $y = 1$, $x = 1 + n$. For $y = 1$ and $x = 1$, it is an explicit identity. We have now to show that if $n1 = 1n$ then $(1 + n)\,1 = 1\,(1 + n)$. Now,

$$
\begin{aligned}
&(1 + n)\,1 \\
&= 1 + n1 && \text{by the definition of multiplication:} \\
&= 1 + 1n && \text{by hypothesis:} \\
&= 1 + n && \text{by the definition of multiplication:} \\
&= 1\,(1 + n) && \text{by the definition of multiplication.}
\end{aligned}
$$

Having thus shown the commutative principle to be true for $y = 1$, we proceed to prove that if it is true for $y = n$, it is true for $y = 1 + n$; that is, if $xn = nx$, then $x(1 + n) = (1 + n)\,x$. For

$$
\begin{aligned}
&(1 + n)\,x \\
&= x + nx && \text{by the definition of multiplication:} \\
&= x + xn && \text{by hypothesis:} \\
&= 1x + xn && \text{by the definition of multiplication:} \\
&= x1 + xn && \text{as already seen:} \\
&= x\,(1 + n) && \text{by the distributive principle.}
\end{aligned}
$$

Discrete Simple Quantity Infinite in both directions.

A system of number infinite in both directions has no minimum, but a certain quantity is called *one*, and the numbers as great as this constitute a partial system of semi-infinite number, of which this one is a minimum. The definitions of addition and multiplication require no change, except that the *one* therein is to be understood in the new sense.

To extend the proofs of the principles of addition and multiplication to nlimited number, it is necessary to show that if true for any number $(1+n)$ hey are also true for the next smaller number n. For this purpose we can se the same transformations as in the second clauses of the former proof; only re shall have to make use of the following lemma.

If $x+y=x+z$, then $y=z$ whatever numbers x, y, and z, may be. 'irst this is true in case $x=1$, for then y and z are both next smaller than the ame number. Therefore, neither is smaller than the other, otherwise it would ot be next smaller to $1+y=1+z$. But in a simple system, of any two ifferent numbers one is smaller. Hence, y and z are equal. Second, if the roposition is true for $x=n$, it is true for $x=1+n$. For if $(1+n)+y=$ $1+n)+z$, then by the definition of addition $1+(n+y)=1+(n+z)$; whence t would follow that $n+y=n+z$, and, by hypothesis, that $y=z$. Third, if he proposition is true for $x=1+n$, it is true for $x=n$. For if $n+y=n+z$, hen $1+n+y=1+n+z$, because the system is simple. The proposition as thus been proved to be true of 1, of every greater and of every smaller umber, and therefore to be universally true.

An inspection of the above proofs of the principles of addition and multi- lication for semi-infinite number will show that they are readily extended to loubly infinite number by means of the proposition just proved.

The number next smaller than one is called naught, 0. This definition in ymbolic form is $1+0=1$. To prove that $x+0=x$, let x' be the number ext smaller than x. Then,

$x+0$	
$=(1+x')+0$	by the definition of x'
$=(1+0)+x'$	by the principles of addition:
$=1+x'$	by the definition of naught:
$=x$	by the definition of x'.

To prove that $x0=0$. First, in case $x=1$, the proposition holds by the lefinition of multiplication. Next, if true for $x=n$, it is true for $x=1+n$. For

$(1+n)\,0$	
$=1.0+n.0$	by the distributive principle:
$=1.0+0$	by hypothesis:
$=1.0$	by the last theorem:
$=0$	as above.

Third, the proposition, if true for $x = 1 + n$ is true for $x = n$. For, changing the order of the transformations,

$$1.0 + 0 = 1.\ 0 = 0 = (1 + n)\ 0 = 1.\ 0 + n.0.$$

Then by the above lemma, $n.\ 0 = 0$, so that the proposition is proved.

A number which added to another gives naught is called the negative of the latter. To prove that every number greater than naught has a negative. First, the number next smaller than naught is the negative of one; for, by the definition of addition, one plus this number is naught. Second, if any number n has a negative, then the number next greater than n has for its negative the number next smaller than the negative of n. For let m be the number next smaller than the negative of n. Then $n + (1 + m) = 0$.

But $n + (1 + m)$

$= (n + 1) + m$ by the associative principle of addition.

$= (1 + n) + m$ by the commutative principle of addition.

So that $(1 + n) + m = 0$. *Q. E. D.* Hence, every number greater than 0 has a negative, and naught is its own negative.

To prove that $(-x)\ y = -(xy)$. We have

$0 = x + (-x)$ by the definition of the negative:

$0 = 0y = (x + (-x))\ y$ by the last proposition but one:

$0 = xy + (-x)\ y$ by the distributive principle:

$-(xy) = (-x)\ y$ by the definition of the negative.

The negative of the negative of a number is that number. For $x + (-x) = 0$. Whence by the definition of the negative $x = -(-x)$.

Limited Discrete Simple Quantity.

Let such a relative term, c, that whatever is a c of anything is the only c of that thing, and is a c of that thing only, be called a relative of simple correspondence. In the notation of the logic of relatives,

$$c\breve{c} \prec 1, \quad \breve{c}c \prec 1.$$

If every object, s, of a class is in any such relation c, with a number of a semi-infinite discrete simple system, and if, further, every number smaller than a number c'd by an s is itself c'd by an s, then the numbers c'd by the s's are

aid to count them, and the system of correspondence is called a count. In ogical notation, putting g for as great as, and n for a positive integral number,

$$s \prec \breve{c}n \qquad \breve{g}cs \prec cs\,.$$

f in any count there is a maximum counting number, the count is said to be nite, and that number is called the number of the count. Let $[s]$ denote the umber of a count of the s's, then

$$[s] \prec cs \qquad \bar{g}cs \prec \overline{[s]}\,.$$

The relative "identical with" satisfies the definition of a relative of simple orrespondence, and the definition of a count is satisfied by putting "identical vith" for c, and "positive integral number as small as x" for s. In this mode of ounting, the number of numbers as small as x is x.

Suppose that in any count the number of numbers as small as the minimum umber, one, is found to be n. Then, by the definition of a count, every number s small as n counts a number as small as one. But, by the definition of one here is only one number as small as one. Hence, by the definition of single orrespondence, no other number than one counts one. Hence, by the definition f one, no other number than one counts any number as small as one. Hence, y the definition of the count, one is, in every count, the number of numbers s small as one.

If the number of numbers as small as x is in some count y, then the umber of numbers as small as y is in some count x. For if the definition of a imple correspondence is satisfied by the relative c, it is equally satisfied by he relative c'd by.

Since the number of numbers as small as x is in some count y, we have, c eing some relative of simple correspondence,

1st. Every number as small as x is c'd by a number.

2d. Every number as small as a number that is c of a number as small as x s itself c of a number as small as x.

3d. The number y is c of a number as small as x.

4th. Whatever is not as great as a number that is c of a number as small as ; is not y.

Now let c_1 be the converse of c. Then the converse of c_1 is c; whence, ince c satisfies the definition of a relative of simple correspondence, so also does c_1. By the 3d proposition above, every number as small as y is as small as a umber that is c of a number as small as x. Whence, by the 2d proposition,

every number as small as y is c of a number as small as x; and it follows that every number as small as y is c_1'd by a number. It follows further that every number c_1 of a number as small as y is c_1 of something c_1'd by (that is, c_1 being a relative of simple correspondence, is identical with) some number as small as x. Also, "as small as" being a transitive relative, every number as small as a number c of a number as small as y is as small as x. Now by the 4th proposition y is as great as any number that is c of a number as small as x, so that what is not as small as y is not c of a number as small as x; whence whatever number is c'd by a number not as small as y is not a number as small as x. But by the 2d proposition every number as small as x not c'd by a number not as small as y is c'd by a number as small as y. Hence, every number as small as x is c'd by a number as small as y. Hence, every number as small as a number c_1 of a number as small as y is c_1 of a number as small as y. Moreover, since we have shown that every number as small as x is c_1 of a number as small as y, the same is true of x itself. Moreover, since we have seen that whatever is c_1 of a number as small as y is as small as x, it follows that whatever is not as great as a number c_1 of a number as small as y is not as great as a number as small as x; *i. e.* ("as great as" being a transitive relative) is not as great as x, and consequently is not x. We have now shown—

1st, that every number as small as y is c_1'd by a number;

2d, that every number as small as a number that is c_1 of a number as small as y is itself c_1 of a number as small as y;

3d, that the number x is c_1 of a number as small as y; and

4th, that whatever is not as great as a number that is c_1 of a number as small as y is not x.

These four propositions taken together satisfy the definition of the number of numbers as small as y counting up to x.

Hence, since the number of numbers as small as one cannot in any count be greater than one, it follows that the number of numbers as small as any number greater than one cannot in any count be one.

Suppose that there is a count in which the number of numbers as small as $1 + m$ is found to be $1 + n$, since we have just seen that it cannot be 1. In this count, let m' be the number which is c of $1 + n$, and let n' be the number which is c'd by $1 + m$. Let us now consider a relative, e, which differs from c only in excluding the relation of m' to $1 + n$ as well as the relation of $1 + m$ to n'

and in including the relation of m' to n'. Then e will be a relative of single correspondence; for c is so, and no exclusion of relations from a single correspondence affects this character, while the inclusion of the relation of m' to n' leaves m' the only e of n' and an e of n' only. Moreover, every number as small as m is e of a number, since every number except $1+m$ that is c of anything is e of something, and every number except $1+m$ that is as small as $1+m$ is as small as m. Also, every number as small as a number e'd by a number is itself e'd by a number; for every number c'd is e'd except $1+m$, and this is greater than any number e'd. It follows that e is the basis of a mode of counting by which the numbers as small as m count up to n. Thus we have shown that if in any way $1+m$ counts up to $1+n$, then in some way m counts up to n. But we have already seen that for $x=1$ the number of numbers as small as x can in no way count up to other than x. Whence it follows that the same is true whatever the value of x.

If every S is a P, and if the P's are a finite lot counting up to a number as small as the number of S's, then every P is an S. For if, in counting the P's, we begin with the S's (which are a part of them), and having counted all the S's arrive at the number n, there will remain over no P's not S's. For if there were any, the number of P's would count up to more than n. From this we deduce the validity of the following mode of inference:

Every Texan kills a Texan,
Nobody is killed by but one person,
Hence, every Texan is killed by a Texan,

supposing Texans to be a finite lot. For, by the first premise, every Texan killed by a Texan is a Texan killer of a Texan. By the second premise, the Texans killed by Texans are as many as the Texan killers of Texans. Whence we conclude that every Texan killer of a Texan is a Texan killed by a Texan, or, by the first premise, every Texan is killed by a Texan. This mode of reasoning is frequent in the theory of numbers.

NOTE.—It may be remarked that when we reason that a certain proposition, if false of any number, is false of some smaller number, and since there is no number (in a semi-limited system) smaller than every number, the proposition must be true, our reasoning is a mere logical transformation of the reasoning that a proposition, if true for n, is true for $1+n$, and that it is true for 1.

On the Remainder of Laplace's Series.

BY EMORY MCCLINTOCK, *Milwaukee, Wis.*

The remainder of Lagrange's series has been given by Popoff and Zolotareff* in the form $\frac{1}{n!}\mathrm{D}^n\int_z^y (x\phi t+z-t)^n f't\,.\,dt$, where $y=z+x\phi y$, and $\mathrm{D}=\frac{d}{dz}$. The corresponding expression for the remainder of Laplace's series, where $y=f(z+x\phi y)$, is $r_n=\frac{1}{n!}\mathrm{D}^n\int_{fz}^y (x\phi t+z-f^{-1}t)^n f't\,.\,dt$.

To prove this, it is only necessary to perform one of the n differentiations indicated, by the usual rule for the differentiation of definite integrals with variable limits. In this way we find that

$$r_n=r_{n-1}-\frac{1}{n!}\,x^n\mathrm{D}^{n-1}(\phi fz)^n\mathrm{D}ffz,$$

whence, if $m=n-1$

$$r_m=\frac{1}{m+1!}\,x^{m+1}\mathrm{D}^m(\phi fz)^{m+1}\mathrm{D}ffz+r_{m+1}.$$

Since $r_0=\mathrm{f}y-\mathrm{f}fz$, we have at once, by successive substitution,

$$\mathrm{f}y=\mathrm{f}fz+x\phi fz\mathrm{D}ffz+\frac{1}{2!}\,x^2\mathrm{D}(\phi fz)^2\mathrm{D}ffz+\ldots+\frac{1}{n!}\,x^n\mathrm{D}^{n-1}(\phi fz)^n\mathrm{D}ffz+r_n.$$

Comparing this result with Laplace's theorem, we see that we have in r_n an expression for the remainder after $n+1$ terms.

If $fz=z$, or, in other words, if $y=z+x\phi y$, we have M. Popoff's expression for the remainder in Lagrange's theorem. The above method of proof resembles in principle that by which Zolotareff proves the latter expression.

It must not be thought that the reasoning here employed constitutes a proof of Laplace's theorem, for evidence is still needed that $r_\infty=0$ when the series is convergent. In other words, the series is not thus proved to be necessarily the true development of $\mathrm{f}y$. Suppose $s_n=\frac{x^{n+1}-1}{1-x}$, so that

$$s_0=-1=x+x^2+x^3+\ldots+x^n+s_n;$$

this would not furnish the true development of -1 in terms of x.

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*See Williamson's *Integral Calculus*, 3d ed.; *Jahrbuch über die Fortschritte der Math.* VIII; these referring to Popoff (*Comptes Rendus*, 1861, pp. 795-8,) and Zolotareff (*Nouv. Annales*, 2d series, XV, 422-8).

from the limits (expressed in terms of x, $x^{\frac{1}{2}}$ and $\log x$) which he has obtained to the number of prime numbers not exceeding x. The object of what follows is to make a little further advance in the same direction, and to show upon Tchebycheff's own principles that the proposition remains true when ε is conditioned no longer to be inferior to the fraction $\frac{1}{5}$, but to the fraction $\frac{1}{6}+\frac{1}{4642^{\frac{10}{11}}}$, so that the excess above unity (the region so to say of darkness) is scarcely more than five-sixths of what it is for the first named fraction. This conclusion is arrived at by aid exclusively of Tchebycheff's own formulae.

Tchebycheff's method may be regarded as the *first* approximation to the inferior and superior limits of a quantity ψx subject to the conditions

$$Vx > Ax + F\log x,$$

$$Vx < Ax + F_1\log x,$$

where $Vx = \psi x - \psi\frac{x}{6} + \psi\frac{x}{7} - \psi\frac{x}{10}$ etc. (see Serret's *Cours d'Algèbre supérieure*, 4th Ed., Vol. 2, pp. 230–233), and to the further conditions that ψx is not less than $\psi x'$ if $x > x'$, and that $\psi x = 0$ when $x < 1$.

The limits obtained for ψx depend exclusively on these definitions, and would be applicable to any function ψx whatever that satisfied them.

The advance made in this article consists in pursuing the approximation through an indefinite number of steps, so as to bring the superior and inferior limits to ψx continually nearer and nearer to each other as regards the *principal* term (a multiple of x) which enters into each of them: the remaining terms over and above this multiple of x in the expressions for the limits always continue to be positive integer powers of $\log x$, and consequently the ratio of the limits becomes as nearly as we please identical with the ratio of the principal terms (*i. e.* of their coefficients) when x is taken sufficiently great: this ratio as given in the first approximation is $\frac{6}{5}$, but as the approximation is continued continually converges to but never reaches the fraction $\frac{7}{6}+\frac{1}{4642^{\frac{10}{11}}}$.

Such, and such only, is the small but not unimportant contribution here supplied to Tchebycheff's remarkable theory. As no allusion is made to the possibility of this contraction of the limits in a work published so recently as 1879, by an

author so competent as M. Serret, I presume that it has hitherto remained unnoticed; but of this I cannot speak with certainty, inasmuch as it was enough for M. Serret's purpose to obtain for the ratio of the principal terms a number less than 2; that being sufficient for the object he had in view, which was to prove M. Bertrand's celebrated postulate that at least one prime number must be included $\left(\text{for all values of } x \text{ greater than } \frac{7}{2}\right)$ between x and $2x-2$.

Although I might confine myself exclusively to the determination of the limits to ψx which flow from the conditions above given, it is, I think, desirable to supply a brief summary of M. Tchebycheff's method, so as to point out the connexion between the determination of these limits and the limits to "the totality of the prime numbers comprised within a given range." In so doing I shall adopt for the convenience of reference the notation which I find in M. Serret's able exposition of the subject (*Alg. sup.*, Vol. 2, pp. 225–239).

θx stands for the sum of the logarithms of all the *prime* numbers not exceeding x.

$$\psi x = \theta x + \theta x^{\frac{1}{2}} + \theta x^{\frac{1}{3}} + \theta x^{\frac{1}{4}} + \theta x^{\frac{1}{5}} + \ldots$$

$$Tx = \psi x + \psi \frac{x}{2} + \psi \frac{x}{3} + \psi \frac{x}{4} + \psi \frac{x}{5} + \ldots$$

and, as a consequence founded on purely arithmetical considerations, Tx is the sum of the logarithms of *all* the numbers not exceeding x, and therefore, as an easy deduction from Stirling's theorem, it follows that for all values of x superior to unity,

$$Tx < x\log x - x + \frac{1}{2}\log x + \left(\log\sqrt{2\pi} + \frac{1}{12}\right)$$

$$Tx > x\log x - x - \frac{1}{2}\log x + \log\sqrt{2\pi}.$$

If then Vx (a notation not in Serret) be used to denote

$$Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{5} + T\frac{x}{30}$$

$\left(\text{where it should be noticed that } 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{30} = 0\right)$, limits for Vx can be found in which $x\log x$ will not appear, and expressed solely in terms of x and $\log x$: it may in fact be shown that for all values of x superior to unity,

$$Vx > A(x-1) - \frac{5}{2}\log x$$

$$Vx < A(x-1) + \frac{5}{2}\log x$$

where $A = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 - \frac{1}{30}\log 30 = .92129202\ldots$.

The limits actually employed, however, are the slightly wider ones,

$$Vx > Ax - \frac{5}{2}\log x - 1$$

$$Vx < Ax + \frac{5}{2}\log x.$$

If now we take an infinite succession of numbers separable into batches of sixteen, such that every $(i+1)^{\text{th}}$ batch may be got by adding $30i$ to each of the numbers in the first batch, those numbers being

1, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 20, 23, 24, 29, 30

(where it is perhaps worth noticing that leaving out the last number 30, the remaining 15 consist of a middle term 15 and pairs of numbers whose sum is always 30, disposed symmetrically about that middle term), it will readily be seen to follow from the expression for V in terms of the T's and of T in terms of the ψ's, that

$$\begin{array}{l} Vx = \psi x - \psi\frac{x}{6} + \psi\frac{x}{7} - \psi\frac{x}{10} + \psi\frac{x}{11} - \psi\frac{x}{12} + \psi\frac{x}{13} - \psi\frac{x}{15} \\ \quad + \psi\frac{x}{17} - \psi\frac{x}{18} + \psi\frac{x}{19} - \psi\frac{x}{20} + \psi\frac{x}{23} - \psi\frac{x}{24} + \psi\frac{x}{29} - \psi\frac{x}{30} \\ \quad + \psi\frac{x}{31} - \psi\frac{x}{36} \ldots\ldots\ldots\ldots - \psi\frac{x}{45} \\ \quad + \psi\frac{x}{47} - \psi\frac{x}{48} \ldots\ldots\ldots\ldots - \psi\frac{x}{60} \\ \quad + \psi\frac{x}{61} \ldots\ldots\ldots\ldots\ldots \\ \quad + \ldots\ldots\ldots\ldots\ldots\ldots \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

just in the same way as if supposing $\omega x = \psi x - 2\psi\frac{x}{2}$ we should find $\omega x = \psi x - \psi\frac{x}{2} + \psi\frac{x}{3} - \psi\frac{x}{4} + \psi\frac{x}{5} \ldots$; or as if supposing $\Omega x = \psi x - \psi\frac{x}{2} - \psi\frac{x}{3} - \psi\frac{x}{6}$ we should find $\Omega x = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \ldots$

From the limits to which Vx is subject (Vx being now regarded as representing the series of ψ's above written) limits can be found to ψx of the form $mx + R_1(\log x)$, $nx + R_2(\log x)$, where the R's signify rational integer forms of function. In the first approximation, for the inferior and superior limits respectively, $m = A$, $n = 6\frac{A}{5}$; R_1 is a linear and R_2 a quadratic form of function. In the approximation of the i^{th} order m and n will become functions of i, and R_1, R_2 will be of the i^{th} and $(i+1)^{\text{th}}$ orders respectively in $\log x$.

The limits of ψx being supposed to be given (say $\psi' x$ the superior and $\psi_1 x$ the inferior limit), $\psi' x$ will serve as a superior and $\psi_1 x - 2\psi' x^{\frac{1}{2}}$ as an inferior limit to θx. But instead of $\psi' x$ we may use (although not at all necessary for the object in view) the slightly closer limit $\psi' x - \psi_1 x^{\frac{1}{2}}$, which is what M. Serret employs, and equally instead of $\psi_1 x - 2\psi' x^{\frac{1}{2}}$ we might use the slightly closer limit

$$\psi_1 x - \psi' x^{\frac{1}{2}} - \psi' x^{\frac{1}{3}} - \psi' x^{\frac{1}{5}} + \psi_1 x^{\frac{1}{30}},$$

which, probably as leading to calculations needlessly complicated (as regards the object in view), M. Serret does not employ. In any case, following the same notation as before to distinguish the two limits, we shall obtain

$$\theta' x = nx + F(x^{\frac{1}{2}}, \log x)$$

$$\theta_1 x = mx + F'(\ . \ . \ . \ , \log x),$$

where F, F' are rational integer forms of function, and the dots in the F' may be filled in either with $x^{\frac{1}{2}}$ or with $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{1}{5}}$, $x^{\frac{1}{30}}$; and we shall have

$$\theta' x = nx(1 + \varepsilon_x) \qquad \theta_1 x = mx(1 + \eta_x),$$

where ε_x and η_x vanish when $x = \infty$.

To come to our ultimate object, it is obvious that the number of primes between x and $(1+\rho)x$ will be greater than $[\theta_1(1+\rho)x - \theta' x] \div \log x$. It will therefore be greater than $\frac{[m(1+\rho)-n]x + \delta_x}{\log x}$, where $\delta_x = 0$ when $x = \infty$. Hence we may find a value of x so great that the number of primes shall be at least K by finding a number x sufficiently large to make $\theta_1(1+\rho)x - \theta' x - (K-1)\log x > 0$, which it must always be possible to do provided that $m(1+\rho) > n$, *i. e.* that $\rho > \left(\frac{n}{m} - 1\right)$. Hence the importance of diminishing

what I call the asymptotic ratio $\frac{n}{m}$, *i. e.* the ratio of the coefficients in the principal terms of the superior and inferior limit to ψx. That is what I shall now proceed to accomplish, but first it is necessary to establish a certain easy lemma.

Suppose the equation $fx - f\frac{x}{c} = Ax^m$ is to be satisfied; this can be done by writing $fx = A\frac{c^m}{c^m - 1}x$, and in particular if $m = 1$, the only case that the present theory demands, $fx = \frac{c}{c-1}Ax$. Again if the equation $fx - f\frac{x}{c} = P(\log x)^\mu$ is to be satisfied, this may be done by making $fx = P_0(\log x)^{\mu+1} + P_1(\log x)^\mu + P_2(\log x)^{\mu-1} + \ldots + P_\mu \log x$, for since $\log\frac{x}{c} = (\log x - \log c)$, $fx - f\frac{x}{c}$ will then obviously become a function of $\log x$ of the μ^{th} order, which may be identified with $P(\log x)^\mu$ by properly assigning the values of the $(\mu + 1)$ disposable constants $P_0, P_1, P_2, \ldots P_\mu$. In fact the equation might easily (if it were worth while to do so) be turned into an equation of differences, and the general values of the P's be expressed once for all in terms of Bernoulli's numbers for any value of μ. Hence it follows that the equation

$$fx - f\frac{x}{c} = Nx + R_\mu{}^* \log x,$$

where R_μ is a rational integer form of function of the μ^{th} order, may be satisfied by making

$$fx = \frac{c}{c-1}Nx + R_{\mu+1}\log x,$$

where the second term on the right hand side of the equation is a *known* function of $\log x$ of the $(\mu + 1)^{\text{th}}$ order.

Suppose now that the inequality $\psi x - \psi\frac{x}{c} < Nx + R_\mu \log x$, where $c > 1$, is given, and it is desired to extract from this inequality an inferior limit to ψx. It is only necessary to get a solution of the equation $fx - f\frac{x}{c} = Nx + R_\mu \log x$.

* The reader's attention is called to the fact that R_μ is used throughout to denote a *form of function*, and not, like P_μ, a *coefficient*.

We shall then have

$$\psi x - \psi\frac{x}{c} < fx - f\frac{x}{c}$$

$$\psi\frac{x}{c} - \psi\frac{x}{c^2} < f\frac{x}{c} - f\frac{x}{c^2}$$

$$\psi\frac{x}{c^2} - \psi\frac{x}{c^3} < f\frac{x}{c^2} - f\frac{x}{c^3}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and consequently $$fx - f\frac{x}{c^q} > \psi x - \psi\frac{x}{c^q}.$$

If then q be supposed to be taken such that $\frac{x}{c^q}$, say z, lies between 0 and 1, we shall have

$$fx - \psi x > fz,$$

and *à fortiori* $> R_{\mu+1}\log z$ (if N be positive, as is the case throughout the present investigation), where the right hand side of the inequality is a known rational integer function of $\log z$. If then M be a number less than the least value that $R_{\mu+1}\xi$ can assume between the limits $\xi = 0$, $\xi = -\log c$, we shall have $\psi x < fx - M$, and an inferior limit will have been obtained to ψx.

In the first approximation (Serret, p. 234), where $\mu = 1$ and $c = 6$, $R_2\xi = \frac{5}{4\log 6}\xi^2 + \frac{5}{4}\xi$, the minimum value of which is got by taking $2\xi = -\log 6$ or $\xi = -\log\sqrt{6}$ (which happens to lie between the limits of log 1 and $-\log 6$) and gives $M = \frac{-5\log 6}{16}$, so that $\psi x < fx + \frac{5\log 6}{16}$. The actual value employed for the superior limit, as sufficiently near and more convenient for use, is $fx + 1$.

So in the general case we shall have $fx - \psi x > M$ where M is any number less than the least value of $R_{\iota+1}\xi$ for values of ξ lying between 0 and $-\log c$. It may or may not be the absolute minimum of $R_{\iota+1}\xi$ that has to be taken according as the value of ξ which gives this absolute minimum does or does not lie between 0 and $-\log c$. In the latter case it may be either some other minimum, or one of the values of $R_{\iota+1}\xi$ corresponding to the extreme values $\xi = 0$ and $\xi = -\log c$, which might be found by trial. But a method practically better and sufficient for the demands made by the present investigation,

would be to substitute zero in place of any term in the function of ξ of the form $+K\xi^{2m}$ or $-K\xi^{2m+1}$, and for any term of the form $-K\xi^{2m}$ or $+K\xi^{2m+1}$ to substitute $-K(\log c)^{2m}$ and $-K(\log c)^{2m+1}$ respectively.

For instance, in the case just considered we might have written $M = -\frac{5}{4}\log 6$, and the superior limit instead of being $fx+1$ would have been $fx+\frac{5}{4}\log 6$, which would practically have been just as good. With a view to a remark which will subsequently be made it is well to notice that the inequality

$$\psi x - \psi\frac{x}{c} > Nx + R_\mu \log x$$

may also be solved precisely in the same manner, and will give for an inferior limit to ψx (using fx to signify the very same function as before) $fx - M_1$, where (N being supposed positive) $M_1 = -N\log c +$ any quantity not less than the greatest value of a known rational integer function of a variable conditioned to lie between 0 and $-\log c$, which may either be found by an exact algebraical process or by substituting 0 in those two cases where previously $-\log c$, and $-\log c$ in those other two cases where previously 0 was to be substituted for the variable.

The lemma needful for our purposes may now accordingly be stated in the following terms: *If* $\psi x - \psi\frac{x}{c}$ *is less or greater than* $Nx +$ *a given rational integer function of* $\log x$ *of any given order*, ψx is less or greater than $\frac{c}{c-1}Nx +$ a known (and easily determinable) rational integer function of $\log x$ of the order next superior.

If the coefficients of x in the superior and inferior limit to ψx at any stage of the investigation be called u and v, I shall show that these values will serve to give (step by step) other superior and inferior limits where u and v are replaced by quantities u', v', such that $u' < u$, $v' > v$; u', v' being known linear functions of u, v. We shall thus be led to a system of two simultaneous linear equations of differences in order to obtain the effect of those changes repeated any number, finite or infinite, of times: but for greater clearness I shall begin with supposing that one of the two expressions u, v, viz. v (which undergoes far less modification than the other) is kept constant. There will then result a single scheme of successive substitutions leading to the construction of a single linear equation in differences.

The first step will then be as follows:

$$\psi x - \psi\frac{x}{6} < Ax + \frac{5}{2}\log x - \psi\frac{x}{7} + \psi\frac{x}{10}$$

$$< Ax + \frac{5}{2}\log x - \left(A\frac{x}{7} - \frac{5}{2}\log\frac{x}{7} - 1\right) + \frac{6}{50}Ax + \frac{5}{4\log 6}\left(\log\frac{x}{10}\right)^2 + \frac{5}{4}\log\frac{x}{10}$$

or writing

$$\lambda = \log 6, \ \mu = \log 7, \ \nu = \log 10,$$

$$\psi x - \psi\frac{x}{6} < \frac{171}{175}Ax + \frac{5}{4\lambda}(\log x)^2 + \left(\left(\frac{25}{4} - \frac{5\nu}{2\lambda}\right)\log x + \frac{5\nu^2}{4\lambda} - \frac{5}{2}\mu - \frac{5}{4}\nu + 1\right).$$

Hence
$$\psi x < \frac{1026}{875}Ax + P(\log x)^3 + Q(\log x)^2 + Rx - M,$$

where first to find P, Q, R, we have the three equations

$$3P\lambda = \frac{5}{4\lambda}$$

$$-3P\lambda^2 + 2Q\lambda = \frac{25}{4} - \frac{5\nu}{2\lambda}$$

$$P\lambda^3 - Q\lambda^2 + R\lambda = \frac{5\nu^2}{4\lambda} - \frac{5}{2}\mu - \frac{5}{4}\nu + 1$$

i. e.
$$P = \frac{5}{12\lambda^2}; \quad Q = \frac{15}{4\lambda} - \frac{5\nu}{4\lambda^2}; \quad R = -\frac{5}{12} + \frac{15}{4} - \frac{5\nu}{2\lambda} + \frac{5\nu^2}{4\lambda^2} - \frac{5\mu}{2\lambda} + \frac{1}{\lambda}.$$

Here P is positive; Q, whose sign depends on that of $3 - \dfrac{\log 10}{\log 6}$, is also positive;

and
$$R = \frac{10}{3} + 5\left(\frac{\nu}{2\lambda} - \frac{1}{2}\right)^2 - \frac{5\mu - 2}{2\lambda} - \frac{5}{4}$$

$$= 3.33333\ldots + .10160\ldots - 2.1570\ldots - 1.25$$
$$= 3.43493\ldots - 3.4070\ldots, \text{ which is also positive.}$$

Hence we may make

$$M = -P\lambda^3 - R\lambda,$$

or
$$-M = 1 + \frac{15}{4}\lambda - \frac{5(\mu + \nu)}{2} + \frac{5\nu^2}{4\lambda} = 1.2947.$$

It is quite possible, and even most likely, that the minimum of $P\lambda^3 - Q\lambda^2 + R\lambda$ (within the prescribed limits) would be found to exceed -1 were it worth while to go through the arithmetical calculations necessary to obtain it, but it is

quite sufficiently near for all practical purposes to use the value above determined; or even to take $-M$ as great as 2 and to adopt for our new superior limit $\frac{171}{175}Ax + P(\log x)^3 + Q(\log x)^2 + R\log x + 2$.

In like manner this new limit will enable us to find another, and it is obvious that the general form of the limit obtained after i of these steps have been gone through will be $u_i Ax + R_{i+2}\log x$, where

$$u_i = \frac{6}{5}\left(1 - \frac{1}{7} + \frac{u_{i-1}}{10}\right) \qquad i.\ e.\quad u_i - \frac{3\,u_{i-1}}{25} = \frac{36}{35}.$$

Putting

$$u_i = \omega_i + h$$

and making

$$\frac{22}{25}h = \frac{36}{35}, \qquad i.\ e.\quad h = \frac{90}{77},$$

we have

$$\omega_i - \frac{3}{25}\omega_{i-1} = 0.$$

Hence

$$u_i = C\left(\frac{3}{25}\right)^i + \frac{90}{77}.$$

The ultimate value of u_i is therefore $\frac{90}{77}$, and accordingly, by repeating the process indicated a sufficient number of times, we shall have for a superior limit $\left(\frac{90}{77} - \varepsilon_i\right)Ax + R_{i+2}\log x$, where ε_i may be made as small as we please by taking i sufficiently great, and thus the ultimate asymptotic ratio of the two limits is $\frac{90}{77}$ instead of $\frac{6}{5}$.

Another mode of approximation may be used, as shown in what follows. Since

$$\psi x - \psi\frac{x}{10} < Ax - \psi\frac{x}{6} + \psi\frac{x}{7};$$

if we have found

$$\psi x < u_i' Ax + R_{i+2}\log x$$

we shall have

$$\psi x - \psi\frac{x}{10} < Ax + u_i' A\frac{x}{6} - A\frac{x}{7} + R_{i+2}'\log x,$$

and therefore

$$\psi x < u'_{i+1} Ax + R_{i+3} \log x,$$

where

$$u'_{i+1} = \frac{10}{9}\left\{1 - \frac{1}{7} + \frac{1}{6}u'_i\right\}$$

i. e.

$$u'_{i+1} - \frac{5}{27}u'_i = \frac{20}{21};$$

or

$$u'_i = K\left(\frac{5}{27}\right)^i + h'$$

where

$$h' = \frac{27}{22}\cdot\frac{20}{21} = \frac{90}{77}.$$

Thus $h' = h$ and consequently also, if we suppose each of the two sorts of approximation to start from the same point, $K = C$.

Hence the ultimate value of u_i and u'_i is the same, but the former method of approximation is to be preferred, as the same number of steps, *i. e.* the same value of i, makes $C\left(\frac{5}{27}\right)^i + h$ always $> C\left(\frac{3}{25}\right)^i + h$. The corresponding values of u_i, u'_i have the same initial and final values, but for every intermediate value of i, $u_i < u'_i$. In fact u_i, u'_i are ordinates to the same abscissa of two non-intersecting curves, having a common starting point and a common asymptote.

The maximum value of $u'_i - u_i$ is found by making $\left(\frac{5}{27}\right)^i - \left(\frac{3}{25}\right)^i$ a maximum, which takes place when i is the integer next above or next below the value

$$\frac{\log\log\frac{25}{3} - \log\log\frac{27}{5}}{\log\frac{25}{3} - \log\frac{27}{5}},$$ which is obviously less than unity.

Hence after the *first* approximation u_i and u'_i are always drawing closer together.

We may now proceed to the more (but only very slightly more) advantageous method of approximation, viz. that in which the principal terms in both limits are simultaneously varied, decreasing as before in the superior, and now at the same time increasing in the inferior limit.

Suppose then that we have found

$$\psi x < u_i\, Ax + R_{i+2} \log x$$
$$\psi x > v_i\, Ax + R_{i+1} \log x;$$

observing that $\frac{v_i}{24} - \frac{u_i}{29}$ is always positive, we shall succeed in increasing the principal term of the inferior limit by writing $\psi x > Ax + v_i\, A\frac{x}{24} - u_i\, A\frac{x}{29} + R_{i+2} \log x$, and slightly more than previously diminishing the principal term in the superior limit by writing $\psi x - \psi\frac{x}{6} < Ax - v_i A\frac{x}{7} + u_i A\frac{x}{10} + R'_{i+2} \log x$. We shall thus easily derive

$$\psi x > v_{i+1}\, Ax + R_{i+2} \log x$$
$$\psi x < u_{i+1}\, Ax + R_{i+3} \log x$$

where

$$v_{i+1} = 1 + \frac{v_i}{24} - \frac{u_i}{29}$$

$$u_{i+1} = \frac{6}{5}\left(1 - \frac{v_i}{7} + \frac{u_i}{10}\right) = \frac{6}{5} - \frac{6}{35} v_i + \frac{3}{25} u_i$$

or, making $v_i = v'_i + f,\ u_i = u'_i + e,$

$$v'_{i+1} - \frac{1}{24} v'_i + \frac{1}{29} u'_i = 0$$

$$u'_{i+1} - \frac{3}{25} u'_i + \frac{6}{35} v'_i = 0,$$

if $\quad \frac{23}{24} f + \frac{1}{29} e = 1, \quad \frac{6}{35} f + \frac{22}{25} e = \frac{6}{5}.$

So that calling $\rho_1,\ \rho_2$ the roots of $\begin{vmatrix} \rho - \frac{1}{24}; & \frac{1}{29} \\ \frac{6}{35}; & \rho - \frac{3}{25} \end{vmatrix} = 0$

$$u_i = C_1\, \rho_1^i + C_2\, \rho_2^i + e$$
$$v_i = C'_1\, \rho_1^i + C'_2\, \rho_2^i + f.$$

The equation for finding $\rho_1,\ \rho_2$ is

$$\rho^2 - \frac{97}{600}\rho - \frac{37}{40600} = 0$$

whence

$$\rho_1 = .167253\ldots \qquad \rho_2 = .005637\ldots$$

Also the equations in e, f give e, f (the values of u_∞, v_∞) as follows:

$$e = \frac{59595}{50999} \qquad f = \frac{51072}{50999}$$

If there were any use in obtaining the values of the disposable constants they could of course be obtained from the equations

$$C_1 + C_2 + e = u_0 = \frac{6}{5} \qquad C_1\rho_1 + C_2\rho_2 + e = u_1 = \frac{1026}{875}$$

$$C_1' + C_2' + f = v_0 = 1 \qquad C_1'\rho_1 + C_2'\rho_2 + f = v_1 = \frac{3481}{3480}$$

The asymptotic ratio of the two limits is

$$\frac{e}{f} = \frac{59595}{51072} = \frac{6}{7} + \frac{11}{51072}$$

Various other modes of approximation may be adopted, but it will be found that no smaller value can be obtained for the asymptotic ratio than that above given: the value of u_∞ cannot be made less than $\frac{59595}{50999}$, nor the value of v_∞ greater than $\frac{51072}{50999}$.

Thus *ex. gr.* making use of the inequality

$$\psi x - \psi\frac{x}{6} > Ax - \psi\frac{x}{7} + \psi\frac{x}{24} - \psi\frac{x}{29} + R\,(\log x),$$

we might by the lemma obtain

$$\psi x > \frac{6}{5}A\left(1 - \frac{u_i}{7} - \frac{u_i}{29} + \frac{v_i}{24}\right) + R_{i+3}\log x,$$

and consequently

$$v_{i+1} = \frac{6}{5}\left(1 - \frac{u_i}{7} - \frac{u_i}{29} + \frac{v_i}{24}\right);$$

combining which with the previous equation for u_{i+1} we should have for finding u_∞, v_∞ say e', f' the two equations,

$$\frac{19}{24}f' + \frac{36}{203}e' = 1$$

$$\frac{6}{35}f' + \frac{22}{25}e' = \frac{6}{5}\,.$$

and consequently

$$e' = \frac{331905}{284029} \qquad f' = \frac{284424}{284069}.$$

Reduced to decimals

$$e = 1.16855\ldots \quad f = 1.00143\ldots$$
$$e' = 1.16856\ldots \quad f' = 1.00125\ldots$$

It may be noticed that $eA = 1.006774\ldots$, $fA = .992619\ldots$ of which the sum is nearly 1.999394, and their mean nearly .999697, whereas the mean of A and $\frac{6A}{5}$ (the original coefficients of x in the limits) is nearly 1.01342. Thus the new mean is more than 44 times nearer than the latter to the true asymptotic value deducible from the empirical formula.

Were it desired merely to find superior and inferior limits to ψx in the form obtained in Tchebycheff's method, it would (as already indicated) have been sufficient to have taken for Vx, $Tx - 2T\frac{x}{2}$, which would have led to the inequalities

$$\psi x > (\log 2)\, x + R_1 \log x$$
$$\psi x < 2(\log 2)\, x + R_2 \log x$$

but the asymptotic ratio being here 2, these limits could not have conducted to a proof of M. Bertrand's postulate. If, however, we were to take $Vx = Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{6}$ we should obtain

$$Vx > Bx + R_1 \log x$$
$$Vx < Bx + R_1' \log x$$

where

$$B = \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{6}\log 6 = 1.0114043$$

and

$$Vx = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \ldots,$$

when we should obtain

$$\psi x - \psi\frac{x}{6} < Bx + R_1' \log x$$

$$\psi x < \frac{6}{5}\, Bx + R_2 \log x,$$

and again

$$\psi x + \psi\frac{x}{5} > Bx + R_1 \log x$$

$$\psi x > B\left(1 - \frac{6}{25}\right)x + R_2' \log x$$

Here the asymptotic ratio of the two limits is $\frac{30}{19}$, which being less than 2, the formulae above indicated would suffice to prove M. Bertrand's postulate, and would lead to an equation somewhat simpler in form than that led to by M. Tchebycheff's process, but whose greatest root would be considerably larger than that found by the established method; so that there would be a larger number of verifications of the postulate to be made for the lower numbers: this, however, is really a matter of very trifling importance, as the needful verifications could be made even up to 100,000 if necessary, by throwing a rapid glance over a few leaves of Burckhardt's tables.

It is noticeable that the limits above found by giving Vx the form $Tx - 2T\frac{x}{2}$ are the *only* limits that can be got in such case; no process of successive approximation being here possible, on account of the too close contiguity of the successive denominators in $\psi x - \psi\frac{x}{2} + \psi\frac{x}{3} \dots$

Such, however, would not be the case were we to use Vx to signify $Tx - T\frac{x}{2} - T\frac{x}{3} - T\frac{x}{6}$, and consequently

$$Vx = \psi x + \psi\frac{x}{5} - 2\psi\frac{x}{6} + \psi\frac{x}{7} + \psi\frac{x}{11} - 2\psi\frac{x}{12} + \psi\frac{x}{13} \dots$$

The limits expressed by the inequalities

$$\psi x < u_i\, Bx + \dots$$
$$\psi x > v_i\, Bx + \dots$$

would lead to the narrower limits

$$\psi x < u_{i+1}\, Bx + \dots$$
$$\psi x > v_{i+1}\, Bx + \dots$$

where

$$u_{i+1} = \frac{6}{5}\left(1 - \frac{v_i}{7} + \frac{u_i}{12}\right)$$

$$v_{i+1} = 1 - \frac{u_i}{5} + \frac{v_i}{6} - \frac{u_i}{11}$$

that is to say

$$u_{i+1} - \frac{u_i}{10} + \frac{6v_i}{35} - \frac{6}{5} = 0\,,$$

$$\frac{u_i}{5} + \frac{u_i}{11} + v_{i+1} - \frac{v_i}{6} - 1 = 0\,.$$

Hence, using as before e, f to indicate the ultimate value of u_i, v_i, we should have

$$21e + 4f - 28 = 0\,,$$
$$96e + 275f - 330 = 0\,.$$

and consequently

$$e = \frac{6380}{5391}\,, \qquad f = \frac{4242}{5391}\,,$$

and

$$\frac{e}{f} = \frac{6380}{4242} = \frac{3}{2} + \frac{1}{249\frac{9}{17}}$$

which is the ultimate value of the asymptotic ratio, of which the initial value was $\frac{30}{19}$, that could be found by this method.

In every such kind of series as I have denoted by Vx, it is obvious that the sum of the multiples of x under the sign of ψ in Vx is equal to the coefficient of x in either limit to Vx. Thus *ex. gr.* in Tchebycheff's series, if we take n a multiple of 30, and make $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, the sum of n terms of $1 - \frac{1}{6} + \frac{1}{7} - \frac{1}{10} + \frac{1}{11} \ldots$

$$= \left(1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{30}\right) S_n + \frac{1}{2}\left(\frac{1}{\frac{1}{2}n+1} + \frac{1}{\frac{1}{2}n+2} + \ldots + \frac{1}{n}\right)$$
$$+ \frac{1}{3}\left(\frac{1}{\frac{1}{3}n+1} + \frac{1}{\frac{1}{3}n+2} + \ldots + \frac{1}{n}\right) + \frac{1}{5}\left(\frac{1}{\frac{1}{5}n+1} + \frac{1}{\frac{1}{5}n+2} + \ldots + \frac{1}{n}\right)$$
$$- \frac{1}{30}\left(\frac{1}{\frac{1}{30}n+1} + \frac{1}{\frac{1}{30}n+2} \ldots + \frac{1}{n}\right);$$

and the multiplier of S_n being always 0, it follows that the sum of an infinite number of the consecutive terms $= \frac{1}{2}\log 2 + \frac{1}{3}\log 3 + \frac{1}{5}\log 5 - \frac{1}{30}\log 30$ $= A$.

It may not unreasonably be conjectured that whilst nothing more can be done with the Tchebycheffian Vx, it may be possible to find such other form of function in lieu of it, or such infinite succession of different forms of function, as may either directly or by successive approximation bring the coefficients of x in the two limits as near as we please to one another, at the expense, of course, of proportionally lengthening out the residues, or tails as they might be termed, of the two limits. Could this be done, it is easy to demonstrate that the limit thus continually approached from opposite sides must be unity, as indicated in advance by Legendre's empirical formula. For this purpose it will be sufficient to use the simplest form of Vx, viz. $Tx - 2T\frac{x}{2}$, whence we obtain

$$\psi x - \psi\frac{x}{2} + \psi\frac{x}{3} \ldots . > \log 2 \,.\, x\,(1 + \varepsilon_x)$$
$$\psi x - \psi\frac{x}{2} + \psi\frac{x}{3} \ldots . < \log 2 \,.\, x\,(1 + \eta_x)$$

ε_x, η_x being known logarithmic quantities which vanish when $x = \infty$.

For suppose it possible to prove that

$$\psi x > Q\,(1 - h)\,x + Gx$$
$$\psi x < Q\,(1 + h)\,x + Fx$$

where $\frac{Fx}{x}$, $\frac{Gx}{x}$ may be made as small as we please by taking x sufficiently large, (I mean by taking x greater than some certain value ξ). Then

$$\begin{aligned}(1 + \varepsilon_x)\log 2 \,.\, x &< \psi x - \psi\frac{x}{2} + \psi\frac{x}{3} \ldots\ldots - \psi\frac{x}{2m} \\ &< Q\,(1 + h)\,x\left(1 - \frac{1}{2} + \frac{1}{3} \ldots\ldots - \frac{1}{2m}\right) \\ &\qquad + Fx - F\frac{x}{2} + F\frac{x}{3} \ldots\ldots - F\frac{x}{2m}.\end{aligned}$$

Let ξ be taken so great that for all values of x greater than $\frac{\xi}{2m}$, $\frac{Fx}{x}$ shall be less in absolute numerical value than $\frac{k}{2m}$, where k is an *arbitrary* positive quantity: then, if we take $x > \xi$, the sum of the absolute values of Fx, $F\frac{x}{2}$, $F\frac{x}{3}$, $\ldots\ldots F\frac{x}{2m}$, is less than kx; and *à fortiori*

$$Fx - F\frac{x}{2} + F\frac{x}{3} \ldots\ldots - F\frac{x}{2m} < kx.$$

Therefore

$$Q\,(1+h)\log 2\,.\,x > (1+\varepsilon_x)\log 2\,.\,x - kx\,.$$

Hence, Q being greater than $\dfrac{1+\varepsilon_x}{1+h} - \dfrac{k}{(1+h)\log 2}$, and ε_x, h, k being all three capable of becoming indefinitely small, $1 - Q$ cannot be a finite positive quantity; which amounts to saying that $1 - Q$ cannot be positive.

In precisely the same manner, dealing with the other limit to Vx and stopping in its development at the term $\psi\dfrac{x}{2m+1}$ $\Big($instead of stopping at the term $-\psi\dfrac{x}{2m}\Big)$ it may be proved that $1 - Q$ cannot be negative. Hence $1 - Q$ must be zero, *i. e.* $Q = 1$. *Q. E. D.*

We have thus determined what is the common limit to which the principal term in the superior and in the inferior limit of ψx are bound to approximate, on the supposition of the possibility of formulae being discoverable admitting of the interval between these principal terms being capable of being made as small as we please. But to pronounce with certainty upon the existence of such possibility, we shall probably have to wait until some one is born into the world as far surpassing Tchebycheff in insight and penetration as Tchebycheff has proved himself superior in these qualities to the ordinary run of mankind.

Specimen of a Literal Table for Binary Quantics, otherwise a Partition Table.

BY PROFESSOR CAYLEY.

The Table, commencing 1; b; c, b^2; d, bc, b^3; . . . , is in fact a Partition Table, viz. considering the letters b, c, d, . . . as denoting 1, 2, 3, . . . respectively, it is 1^0; 1; 2, 11; 3, 12, 111; . . . a table of the partitions of the numbers 0, 1, 2, 3 . . . , expressed however in the literal form, in order to its giving the literal terms which enter into the coefficients of any covariant of a binary quantic. The table ought to have been made and published many years ago, before the calculation of the covariants of the quintic; and the present publication of it is, in some measure, an anachronism: but I in fact felt the need of it in some calculations in regard to the sextic; and I think the table may be found useful on other occasions. I have contented myself with calculating the table up to $s = 18$, that is, so as to include in it all the partitions of 18: it would, I think, be desirable to extend it further, say to $z = 26$; or even beyond this point, but perhaps without introducing any new letters, (that is, so as to give for the higher numbers only the partitions with a largest part not exceeding 26): the question of the space which such a table would occupy will be considered presently.

As to the employment of the table, observe that in applying it to the case of a quantic $(a,\ b,\ c,\ d \between x,\ y)^3$, the terms containing the letters e, f, etc., posterior to the last coefficient d of the quantic are to be disregarded; and that the terms are to be rendered homogeneous by the introduction of the proper power of the first coefficient a, rejecting any term for which the exponent of a would be negative (or what is the same thing, any term of too high a degree in

the coefficients b, c, d); thus, for the cubicovariant, where the coefficients are of the degree 3, and of the weights 3, 4, 5, 6 respectively, from the portion of the table

d	e	f	g
bc	bd	be	bf
b^3	c^2	cd	ce
	b^2c	b^2d	d^2
	b^4	bc^2	b^2e
		b^3c	bcd
		b^5	c^3
			b^3d
			etc.

we at once copy out the terms

a^2d	abd	acd	ad^2
abc	ac^2	b^2d	bcd
b^3	b^2c	bc^2	c^3

which compose the coefficients in question.

As regards the formation of the table, this is at once effected, and the successive terms are obtained *currente calamo*, by Arbogast's rule of the last and the last but one: observing that each term is to be regarded as containing implicitly a power of a, so that operating on any term such as b^4, the operation on the last letter gives b^3c, and that on the last but one letter gives b^5. There is little risk of error except in the accidental omission of a term; but of course any one omission would occasion the omission of all the subsequent terms derivable from the omitted term, and would so be fatal: to remove this source of error, observe that for the successive numbers 0, 1, 2, 3, etc., the number of partitions should be

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18 ...
1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385 ...

and we can thus, for each partible number successively, verify that the right number of partitions has been obtained.

But as the number of partitions becomes large, a further control is convenient, and even necessary—say we have the 176 partitions of 15, we have by the rule to derive thence the 231 partitions of 16, and it is not until the whole of this derivation is gone through, that we could by counting the number of the new terms ascertain that the right number of 231 terms has been obtained. To break up the verification, it is convenient to know that for the partitions of 16 into 1 part, 2 parts, 3 parts, 4 parts, etc., the numbers of partitions are 1, 8, 21, 34, etc., respectively: we can then as soon as the derivations giving the partitions

into 1 part, 2 parts, 3 parts, etc., respectively, have been performed, verify that the right numbers 1, 8, 21, 34, etc., of terms have been obtained. The numbers are contained in the following table, each column of which is calculated from the preceding columns according to a rule which is easily obtained, and which is itself verified by the condition that the sums of the numbers in the several columns give the before mentioned series of numbers 1, 1, 2, 3, 5, 7, etc.

No. of	PARTIBLE NUMBER.																		
Parts.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2			1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
3				1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27
4					1	1	2	3	5	6	9	11	15	18	23	27	34	39	47
5						1	1	2	3	5	7	10	13	18	23	30	37	47	57
6							1	1	2	3	5	7	11	14	20	26	35	44	58
7								1	1	2	3	5	7	11	15	21	28	38	49
8									1	1	2	3	5	7	11	15	22	29	40
9										1	1	2	3	5	7	11	15	22	30
10											1	1	2	3	5	7	11	15	22
11												1	1	2	3	5	7	11	15
12													1	1	2	3	5	7	11
13														1	1	2	3	5	7
14															1	1	2	3	5
15																1	1	2	3
16																	1	1	2
17																		1	1
18																			1
	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385

The practical rule for the construction of the table thus is:— On a sheet of paper ruled in squares, and which is read as a continuous column from the bottom of one column to the top of the next column, form the terms by Arbogast's method as already explained; writing down in pencil a batch of terms, and counting them to see that the right number has been obtained, then, at the same time

verifying the derivations, mark these over in ink; and so on with another batch of terms, until the whole number of the partitions of any particular number is obtained.

The foregoing series 1, 1, 2, 3, . . . 385, for the number of the partitions of the successive numbers 0, 1, 2, 3 . . . 18 is carried by Euler up to the number of partitions of 59, = 831820, see the paper *De Partitione Numerorum*, Op. Arith. Coll. I., bottom line of the table pp. 97–101: the continuation from the number 385 and for the partible numbers 19 to 30 is as follows:

19	20	21	22	23	24	25	26	27	28	29	30
490	627	792	1002	1255	1575	1958	2436	3010	3718	4565	5604;

the whole number of terms 1, 1, . . . 5604 amounts to 28629, which at the rate of 500 to a page would occupy somewhat under 60 pages; or, at the rate here employed of 369 to a page, somewhat under 78 pages.

THE PARTITION TABLE, 0 TO 18.

0 · 3	4 · 5	6 · 7	7 · 8	8 · 9	9	9 · 10	10	10 · 11
$\boxed{\begin{matrix}\mathbf{0}\\ 1\end{matrix}}$ 1 $\boxed{\begin{matrix}\mathbf{1}\\ 1\end{matrix}}$ b $\boxed{\begin{matrix}\mathbf{2}\\ 2\end{matrix}}$ c b^2 $\boxed{\begin{matrix}\mathbf{3}\\ 3\end{matrix}}$ d bc b^3	$\boxed{\begin{matrix}\mathbf{4}\\ 5\end{matrix}}$ e bd c^2 b^2c b^4 $\boxed{\begin{matrix}\mathbf{5}\\ 7\end{matrix}}$ f be cd b^2d bc^2 b^3c b^5	$\boxed{\begin{matrix}\mathbf{6}\\ 11\end{matrix}}$ g bf ce d^2 b^2e bcd c^3 b^3d b^2c^2 b^4c b^6 $\boxed{\begin{matrix}\mathbf{7}\\ 15\end{matrix}}$ h bg	cf de b^2f bce bd^2 c^2d b^3e b^2cd bc^3 b^4d b^3c^2 b^5c b^7 $\boxed{\begin{matrix}\mathbf{8}\\ 22\end{matrix}}$ i bh cg df e^2	b^2g bcf bde c^2e cd^2 b^3f b^2ce b^2d^2 bc^2d c^4 b^4e b^3cd b^2c^3 b^5d b^4c^2 b^6c b^8 $\boxed{\begin{matrix}\mathbf{9}\\ 30\end{matrix}}$ j	bi ch dg ef b^2h bcg bdf be^2 c^2f cde d^3 b^3g b^2cf b^2de bc^2e bcd^2 c^3d b^4f b^3ce b^3d^2 b^2c^2d	bc^4 b^5e b^4cd b^3c^3 b^6d b^5c^2 b^7c b^9 $\boxed{\begin{matrix}\mathbf{10}\\ 42\end{matrix}}$ k bj ci dh eg f^2 b^2i bch	bdg bef c^2g cdf ce^2 d^2e b^3h b^2cg b^2df b^2e^2 bc^2f $bcde$ bd^3 c^3e c^2d^2 b^4g b^3cf b^3de b^2c^2e b^2cd^2 bc^3d	c^5 b^5f b^4ce b^4d^2 b^3c^2d b^2c^4 b^6e b^5cd b^4c^3 b^7d b^6c^2 b^8c b^{10} $\boxed{\begin{matrix}\mathbf{11}\\ 56\end{matrix}}$ l bk cj

11	11 . 12	12	12 . 13	13	13 . 14	14	14	14
di	b^6f	$bcef$	b^6d^2	$bceg$	bc^3d^2	gi	b^4k	c^5e
eh	b^5ce	bd^2f	b^5c^2d	bcf^2	c^5d	h^2	b^3cj	c^4d^2
fg	b^5d^2	bde^2	b^4c^4	bd^2g	b^6h	b^2m	b^3di	b^6i
b^2j	b^4c^2d	c^3g	b^8e	$bdef$	b^5cg	bcl	b^3eh	b^5ch
bci	b^3c^4	c^2df	b^7cd	be^3	b^5df	bdk	b^3fg	b^5dg
bdh	b^7e	c^2e^2	b^6c^3	c^3h	b^5e^2	bej	b^2c^2i	b^5ef
beg	b^6cd	cd^2e	b^9d	c^2dg	b^4c^2f	bfi	b^2cdh	b^4c^2g
bf^2	b^5c^3	d^4	b^8c^2	c^2ef	b^4cde	bgh	b^2ceg	b^4cdf
c^2h	b^8d	b^4i	$b^{10}c$	cd^2f	b^4d^3	c^2k	b^2cf^2	b^4ce^2
cdg	b^7c^2	b^3ch	b^{12}	cde^2	b^3c^3e	cdj	b^2d^2g	b^4d^2e
cef	b^9c	b^3dg	**13** 101	d^3e	$b^3c^2d^2$	cei	b^2def	b^3c^3f
d^2f	b^{11}	b^3ef		b^4j	b^2c^4d	cfh	b^2e^3	b^3c^2de
de^2	**12** 77	b^2c^2g		b^3ci	bc^6	cg^2	bc^3h	b^3cd^3
b^3i		b^2cdf	n	b^3dh	b^7g	d^2i	bc^2dg	b^2c^4e
b^2ch		b^2ce^2	bm	b^3eg	b^6cf	deh	bc^2ef	$b^2c^3d^2$
b^2dg	m	b^2d^2e	cl	b^3f^2	b^6de	dfg	bcd^2f	bc^5d
b^2ef	bl	bc^3f	dk	b^2c^2h	b^5c^2e	e^2g	$bcde^2$	c^7
bc^2g	ck	bc^2de	ej	b^2cdg	b^5cd^2	ef^2	bd^3e	b^7h
$bcdf$	dj	bcd^3	fi	b^2cef	b^4c^3d	b^3l	c^4g	b^6cg
bce^2	ei	c^4e	gh	b^2d^2f	b^3c^5	b^2ck	c^3df	b^6df
bd^2e	fh	c^3d^2	b^2l	b^2de^2	b^8f	b^2dj	c^3e^2	b^6e^2
c^3f	g^2	b^5h	bck	bc^3g	b^7ce	b^2ei	c^2d^2e	b^5c^2f
c^2de	b^2k	b^4cg	bdj	bc^2df	b^7d^2	b^2fh	cd^4	b^5cde
cd^3	bcj	b^4df	bei	bc^2e^2	b^6c^2d	b^2g^2	b^5j	b^5d^3
b^4h	bdi	b^4e^2	bfh	bcd^2e	b^5c^4	bc^2j	b^4ci	b^4c^3e
b^3cg	beh	b^3c^2f	bg^2	bd^4	b^9e	$bcdi$	b^4dh	$b^4c^2d^2$
b^3df	bfg	b^3cde	c^2j	c^4f	b^8cd	$bceh$	b^4eg	b^3c^4d
b^3e^2	c^2i	b^3d^3	cdi	c^3de	b^7c^3	$bcfg$	b^4f^2	b^2c^6
b^2c^2f	cdh	$b^2c\ e$	ceh	c^2d^3	$b^{10}d$	bd^2h	b^3c^2h	b^8g
b^2cde	ceg	$b^2c^2d^2$	cfg	b^5i	b^9c^2	$bdeg$	b^3cdg	b^7cf
b^2d^3	cf^2	bc^4d	d^2h	b^4ch	$b^{11}c$	bdf^2	b^3cef	b^7de
bc^3e	d^2g	c^6	deg	b^4dg	b^{13}	be^2f	b^3d^2f	b^6c^2e
bc^2d^2	def	b^6g	df^2	b^4ef	**14** 135	c^3i	b^3de^2	b^6cd^2
c^4d	e^3	b^5cf	e^2f	b^3c^2g		c^2dh	b^2c^3g	b^5c^3d
b^5g	b^3j	b^5de	b^3k	b^3cdf		c^2eg	b^2c^2df	b^4c^5
b^4cf	b^2ci	b^4c^2e	b^2cj	b^3ce^2	o	c^2f^2	$b^2c^2e^2$	b^9f
b^4de	b^2dh	b^4cd^2	b^2di	b^3d^2e	bn	cd^2g	b^2cd^2e	b^8ce
b^3c^2e	b^2eg	b^3c^3d	b^2eh	b^2c^3f	cm	$cdef$	b^2d^4	b^8d^2
b^3cd^2	b^2f^2	b^2c^5	b^2fg	b^2c^2de	dl	ce^3	bc^4f	b^7c^2d
b^2c^3d	bc^2h	b^7f	bc^2i	b^2cd^3	ek	d^3f	bc^3de	b^6c^4
bc^5	$bcdg$	b^6ce	$bcdh$	bc^4e	fj	d^2e^2	bc^2d^3	$b^{10}e$

14 . 15	15	15	15	15 . 16	16	16	16	16
b^9cd	b^2gh	$bcdef$	b^5f^2	b^4c^4d	cdl	cef^2	cd^2e^2	b^5fg
b^8c^3	bc^2k	bce^3	b^4c^2h	b^3c^6	cek	d^3h	d^4e	b^4c^2i
$b^{11}d$	$bcdj$	bd^3f	b^4cdg	b^9g	cfj	d^2eg	b^5l	b^4cdh
$b^{10}c^2$	$bcei$	bd^2e^2	b^4cef	b^8cf	cgi	d^2f^2	b^4ck	b^4ceg
$b^{12}c$	$bcfh$	c^4h	b^4d^2f	b^8de	ch^2	de^2f	b^4dj	b^4cf^2
b^{14}	bcg^2	c^3dg	b^4de^2	b^7c^2e	d^2k	e^4	b^4ei	b^4d^2g
15	bd^2i	c^3ef	b^3c^3g	b^7cd^2	dej	b^4m	b^4fh	b^4def
176	$bdeh$	c^2d^2f	b^3c^2df	b^6c^3d	dfi	b^3cl	b^4g^2	b^4e^3
	$bdfg$	c^2de^2	$b^3c^2e^2$	b^5c^5	dgh	b^3dk	b^3c^2j	b^3c^3h
p	be^2g	cd^3e	b^3cd^2e	$b^{10}f$	e^2i	b^3ej	b^3cdi	b^3c^2dg
bo	bef^2	d^5	b^3d^4	b^9ce	efh	b^3fi	b^3ceh	b^3c^2ef
cn	c^3j	b^5k	b^2c^4f	b^9d^2	eg^2	b^3gh	b^3cfg	b^3cd^2f
dm	c^2di	b^4cj	b^2c^3de	b_8c^2d	f^2g	b^2c^2k	b^3d^2h	b^3cde^2
el	c^2eh	b^4di	$b^2c^2d^3$	b^7c^4	b^3n	b^2cdj	b^3deg	b^3d^3e
fk	c^2fg	b^4eh	bc^5e	$b^{11}e$	b^2cm	b^2cei	b^3df^2	b^2c^4g
gj	cd^2h	b^4fg	bc^4d^2	$b^{10}cd$	b^2dl	b^2cfh	b^3e^2f	b^2c^3df
hi	$cdeg$	b^3c^2i	c^6d	b^9c^3	b^2ek	b^2cg^2	b^2c^3i	$b^2c^3e^2$
b^2n	cdf^2	b^3cdh	b^7i	$b^{12}d$	b^2fj	b^2d^2i	b^2c^2dh	$b^2c^2d^2e$
bcm	ce^2f	b^3ceg	b^6ch	$b^{11}c^2$	b^2gi	b^2deh	b^2c^2eg	b^2cd^4
bdl	d^3g	b^3cf^2	b^6dg	$b^{13}c$	b^2h^2	b^2dfg	$b^2c^2f^2$	bc^5f
bek	d^2ef	b^3d^2g	b^6ef	b^{15}	bc^2l	b^2e^2g	b^2cd^2g	bc^4de
bfj	de^3	b^3def	b^5c^2g	**16**	$bcdk$	b^2ef^2	b^2cdef	bc^3d^3
bgi	b^4l	b^3e^3	b^5cdf	231	$bcej$	bc^3j	b^2ce^3	c^6e
bh^2	b^3ck	b^2c^3h	b^5ce^2		$bcfi$	bc^2di	b^2d^3f	c^5d^2
c^2l	b^3dj	b^2c^2dg	b^5d^2e	q	$bcgh$	bc^2eh	$b^2d^2e^2$	b^7j
cdk	b^3ei	b^2c^2ef	b^4c^3f	bp	bd^2j	bc^2fg	bc^4h	b^6ci
cej	b^3fh	b^2cd^2f	b^4c^2de	co	$bdei$	bcd^2h	bc^3dg	b^6dh
cfi	b^3g^2	b^2cde^2	b^4cd^3	dn	$bdfh$	$bcdeg$	bc^3ef	b^6eg
cgh	b^2c^2j	b^2d^3e	b^3c^4e	em	bdg^2	$bcdf^2$	bc^2d^2f	b^6f^2
d^2j	b^2cdi	bc^4g	$b^3c^3d^2$	fl	be^2h	bce^2f	bc^2de^2	b^5c^2h
dei	b^2ceh	bc^3df	b^2c^5d	gk	$befg$	bd^3g	bcd^3e	b^5cdg
dfh	b^2cfg	bc^3e^2	bc^7	hj	bf^3	bd^2ef	bd^5	b^5cef
dg^2	b^2d^2h	bc^2d^2e	b^8h	i^2	c^3k	bde^3	c^5g	b^5d^2f
e^2h	b^2deg	bcd^4	b^7cg	b^2o	c^2dj	c^4i	c^4df	b^5de^2
efg	b^2df^2	c^5f	b^7df	bcn	c^2ei	c^3dh	c^4e^2	b^4c^3g
f^3	b^2e^2f	c^4de	b^7e^2	bdm	c^2fh	c^3eg	c^3d^2e	b^4c^2df
b^3m	bc^3i	c^3d^3	b^6c^2f	bel	c^2g^2	c^3f^2	c^2d^4	$b^4c^2e^2$
b^2cl	bc^2dh	b^6j	b^6cde	bfk	cd^2i	c^2d^2g	b^6k	b^4cd^2e
b^2dk	bc^2eg	b^5ci	b^6d^3	bgj	$cdeh$	c^2def	b^5cj	b^4d^4
b^2ej	bc^2f^2	b^5dh	b^5c^3e	bhi	$cdfg$	c^2e^3	b^5di	b^3c^4f
b^2fi	bcd^2g	b^5eg	$b^5c^2d^2$	c^2m	ce^2g	cd^3f	b^5eh	b^3c^3de

16	16 : 17	17	17	17	17	17	17	17 . 18
$b^3c^2d^3$	b^9c^2d	efi	def^2	c^2deg	bc^2e^3	bc^5g	b^7f^2	$b^7c^2d^2$
b^2c^5e	b^8c^4	egh	e^3f	c^2df^2	bcd^3f	bc^4df	b^6c^2h	b^6c^4d
$b^2c^4d^2$	$b^{12}e$	f^2h	b^4n	c^2e^2f	bcd^2e^2	bc^4e^2	b^6cdg	b^5c^6
bc^6d	$b^{11}cd$	fg^2	b^3cm	cd^3g	bd^4e	bc^3d^2e	b^6cef	$b^{11}g$
c^8	$b^{10}c^3$	b^3o	b^3dl	cd^2ef	c^5h	bc^2d^4	b^6d^2f	$b^{10}cf$
b^8i	$b^{13}d$	b^2cn	b^3ek	cde^3	c^4dg	c^6f	b^6de^2	$b^{10}de$
b^7ch	$b^{12}c^2$	b^2dm	b^3fj	d^4f	c^4ef	c^5de	b^5c^3g	b^9c^2e
b^7dg	$b^{14}c$	b^2el	b^3gi	d^3e^2	c^3d^2f	c^4d^3	b^5c^2df	b^9cd^2
b^7ef	b^{16}	b^2fk	b^3h^2	b^5m	c^3de^2	b^7k	$b^5c^2e^2$	b^8c^3d
b^6c^2g	**17**	b^2gj	b^2c^2l	b^4cl	c^2d^3e	b^6cj	b^5cd^2e	b^7c^5
b^6cdf	297	b^2hi	b^2cdk	b^4dk	cd^5	b^6di	b^5d^4	$b^{12}f$
b^6ce^2		bc^2m	b^2cej	b^4ej	b^6l	b^6eh	b^4c^4f	$b^{11}ce$
b^6d^2e	r	$bcdl$	b^2cfi	b^4fi	b^5ck	b^6fg	b^4c^3de	$b^{11}d^2$
b^5c^3f	bq	$bcek$	b^2cgh	b^4gh	b^5dj	b^5c^2i	$b^4c^2d^3$	$b^{10}c^2d$
b^5c^2de	cp	$bcfj$	b^2d^2j	b^3c^2k	b^5ei	b^5cdh	b^3c^5e	b^9c^4
b^5cd^3	do	$bcgi$	b^2dei	b^3cdj	b^5fh	b^5ceg	$b^3c^4d^2$	$b^{13}e$
b^4c^4e	en	bch^2	b^2dfh	b^3cei	b^5g^2	b^5cf^2	b^2c^6d	$b^{12}cd$
$b^4c^3d^2$	fm	bd^2k	b^2dg^2	b^3cfh	b^4c^2j	b^5d^2g	bc^8	$b^{11}c^3$
b^3c^5d	gl	$bdej$	b^2e^2h	b^3cg^2	b^4cdi	b^5def	b^9i	$b^{14}d$
b^2c^7	hk	$bdfi$	b^2efg	b^3d^2i	b^4ceh	b^5e^3	b^8ch	$b^{13}c^2$
b^9h	ij	$bdgh$	b^2f^3	b^3deh	b^4cfg	b^4c^3h	b^8dg	$b^{15}c$
b^8cg	b^2p	be^2i	bc^3k	b^3dfg	b^4d^2h	b^4c^2dg	b^8ef	b^{17}
b^8df	bco	$befh$	bc^2dj	b^3e^2g	b^4deg	b^4c^2ef	b^7c^2g	
b^8e^2	bdn	beg^2	bc^2ei	b^3ef^2	b^4df^2	b^4cd^2f	b^7cdf	**18**
b^7c^2f	bem	bf^2g	bc^2fh	b^2c^3j	b^4e^2f	b^4cde^2	b^7ce^2	385
b^7cde	bfl	c^3l	bc^2g^2	b^2c^2di	b^3c^3i	b^4d^3e	b^7d^2e	s
b^7d^3	bgk	c^2dk	bcd^2i	b^2c^2eh	b^3c^2dh	b^3c^4g	b^6c^3f	br
b^6c^3e	bhj	c^2ej	$bcdeh$	b^2c^2fg	b^3c^2eg	b^3c^3df	b^6c^2de	cq
$b^6c^2d^2$	bi^2	c^2fi	$bcdfg$	b^2cd^2h	$b^3c^2f^2$	$b^3c^3e^2$	b^6cd^3	dp
b^5c^4d	c^2n	c^2gh	bce^2g	b^2cdeg	b^3cd^2g	$b^3c^2d^2e$	b^5c^4e	eo
b^4c^6	cdm	cd^2j	$bcef^2$	b^2cdf^2	b^3cdef	b^3cd^4	$b^5c^3d^2$	fn
$b^{10}g$	cel	$cdei$	bd^3h	b^2ce^2f	b^3ce^3	b^2c^5f	b^4c^5d	gm
b^9cf	cfk	$cdfh$	bd^2eg	b^2d^3g	b^3d^3f	b^2c^4de	b^3c^7	hl
b^9de	cgj	cdg^2	bd^2f^2	b^2d^2ef	$b^3d^2e^2$	$b^2c^3d^3$	$b^{10}h$	ik
b^8c^2e	chi	ce^2h	bde^2f	b^2de^3	b^2c^4h	bc^6e	b^9cg	j^2
b^8cd^2	d^2l	$cefg$	be^4	bc^4i	b^2c^3dg	bc^5d^2	b^9df	b^2q
b^7c^3d	dek	cf^3	c^4j	bc^3dh	b^2c^3ef	c^7d	b^9e^2	bcp
b^6c^5	dfj	d^3i	c^3di	bc^3eg	$b^2c^2d^2f$	b^8j	b^8c^2f	bdo
$b^{11}f$	dgi	d^2eh	c^3eh	bc^3f^2	$b^2c^2de^2$	b^7ci	b^8cde	ben
$b^{10}ce$	dh^2	d^2fg	c^3fg	bc^2d^2g	b^2cd^3e	b^7dh	b^8d^3	bfm
$b^{10}d^2$	e^2j	de^2g	c^2d^2h	bc^2def	b^2d^5	b^7eg	b^7c^3e	bgl

18	18	18	18	18	18	18	18	18
bhk	$befi$	b^2dej	d^3ef	bc^2deg	b^3cdf^2	b^4c^3i	b^5cd^2f	b^8c^2g
bij	$begh$	b^2dfi	d^2e^3	bc^2df^2	b^3ce^2f	b^4c^2dh	b^5cde^2	b^8cdf
c^2o	bf^2h	b^2dgh	b^5n	bc^2e^2f	b^3d^3g	b^4c^2eg	b^5d^3e	b^8ce^2
cdn	bfg^2	b^2e^2i	b^4cm	bcd^3g	b^3d^2ef	$b^4c^2f^2$	b^4c^4g	b^8d^2e
cem	c^3m	b^2efh	b^4dl	bcd^2ef	b^3de^3	b^4cd^2g	b^4c^3df	b^7c^3f
cfl	c^2dl	b^2eg^2	b^4ek	$bcde^3$	b^2c^4i	b^4cdef	$b^4c^3e^2$	b^7c^2de
cgk	c^2ek	b^2f^2g	b^4fj	bd^4f	b^2c^3dh	b^4ce^3	$b^4c^2d^2e$	b^7cd^3
chj	c^2fj	bc^3l	b^4gi	bd^3e^2	b^2c^3eg	b^4d^3f	b^4cd^4	b^6c^4e
ci^2	c^2gi	bc^2dk	b^4h^2	c^5i	$b^2c^3f^2$	$b^4d^2e^2$	b^3c^5f	$b^6c^3d^2$
d^2m	c^2h^2	bc^2ej	b^3c^2l	c^4dh	$b^2c^2d^2g$	b^3c^4h	b^3c^4de	b^5c^5d
del	cd^2k	bc^2fi	b^3cdk	c^4eg	b^2c^2def	b^3c^3dg	$b^3c^3d^3$	b^4c^7
dfk	$cdej$	bc^2gh	b^3cej	c^4f^2	$b^2c^2e^3$	b^3c^3ef	b^2c^6e	$b^{11}h$
dgj	$cdfi$	bcd^2j	b^3cfi	c^3d^2g	b^2cd^3f	$b^3c^2d^2f$	$b^2c^5d^2$	$b^{10}cg$
dhi	$cdgh$	$bcdei$	b^3cgh	c^3def	$b^2cd^2e^2$	$b^3c^2de^2$	bc^7d	$b^{10}df$
e^2k	ce^2i	$bcdfh$	b^3d^2j	c^3e^3	b^2d^4e	b^3cd^3e	c^9	$b^{10}e^2$
efj	$cefh$	$bcdg^2$	b^3dei	c^2d^3f	bc^5h	b^3d^5	b^9j	b^9c^2f
egi	ceg^2	bce^2h	b^3dfh	$c^2d^2e^2$	bc^4dg	b^2c^5g	b^8ci	b^9cde
eh^2	cf^2g	$bcefg$	b^3dg^2	cd^4e	bc^4ef	b^2c^4df	b^8dh	b^9d^3
f^2i	d^3j	bcf^3	b^3e^2h	d^6	bc^3d^2f	$b^2c^4e^2$	b^8eg	b^8c^3e
fgh	d^2ei	bd^3i	b^3efg	b^6m	bc^3de^2	$b^2c^3d^2e$	b^8f^2	$b^8c^2d^2$
g^3	d^2fh	bd^2eh	b^3f^3	b^5cl	bc^2d^3e	$b^2c^2d^4$	b^7c^2h	b^7c^4d
b^3p	d^2g^2	bd^2fg	b^2c^3k	b^5dk	bcd^5	bc^6f	b^7cdg	b^6c^6
b^2co	de^2h	bde^2g	b^2c^2dj	b^5ej	c^6g	bc^5de	b^7cef	$b^{12}g$
b^2dn	$defg$	$bdef^2$	b^2c^2ei	b^5fi	c^5df	bc^4d^3	b^7d^2f	$b^{11}cf$
b^2em	df^3	be^3f	b^2c^2fh	b^5gh	c^5e^2	c^7e	b^7de^2	$b^{11}de$
b^2fl	e^3g	c^4k	$b^2c^2g^2$	b^4c^2k	c^4d^2e	c^6d^2	b^6c^3g	$b^{10}c^2e$
b^2gk	e^2f^2	c^3dj	b^2cd^2i	b^4cdj	c^3d^4	b^8k	b^6c^2df	$b^{10}cd^2$
b^2hj	b^4o	c^3ei	b^2cdeh	b^4cei	b^7l	b^7cj	$b^6c^2e^2$	b^9c^3d
b^2i^2	b^3cn	c^3fh	b^2cdfg	b^4cfh	b^6ck	b^7di	b^6cd^2e	b^8c^5
bc^2n	b^3dm	c^3g^2	b^2ce^2g	b^4cg^2	b^6dj	b^7eh	b^6d^4	$b^{13}f$
$bcdm$	b^3el	c^2d^2i	b^2cef^2	b^4d^2i	b^6ei	b^7fg	b^5c^4f	$b^{12}ce$
$bcel$	b^3fk	c^2deh	b^2d^3h	b^4deh	b^6fh	b^6c^2i	b^5c^3de	$b^{12}d^2$
$bcfk$	b^3gj	c^2dfg	b^2d^2eg	b^4dfg	b^6g^2	b^6cdh	$b^5c^2d^3$	$b^{11}c^2d$
$bcgj$	b^3hi	c^2e^2g	$b^2d^2f^2$	b^4e^2g	b^5c^2j	b^6ceg	b^4c^5e	$b^{10}c^4$
$bchi$	b^2c^2m	c^2ef^2	b^2de^2f	b^4ef^2	b^5cdi	b^6cf^2	$b^4c^4d^2$	$b^{14}e$
bd^2l	b^2cdl	cd^3h	b^2e^4	b^3c^3j	b^5ceh	b^6d^2g	b^3c^6d	$b^{13}cd$
$bdek$	b^2cek	cd^2eg	bc^4j	b^3c^2di	b^5cfg	b^6def	b^2c^8	$b^{12}c^3$
$bdfj$	b^2cfj	cd^2f^2	bc^3di	b^3c^2eh	b^5d^2h	b^6e^3	$b^{10}i$	$b^{15}d$
$bdgi$	b^2cgi	cde^2f	bc^3eh	b^3c^2fg	b^5deg	b^5c^3h	b^9ch	$b^{14}c^2$
bdh^2	b^2ch^2	ce^4	bc^3fg	b^3cd^2h	b^5df^2	b^5c^2dg	b^9dg	$b^{16}c$
be^2j	b^2d^2k	d^4g	bc^2d^2h	b^3cdeg	b^5e^2f	b^5c^2ef	b^9ef	b^{18}

Note on Hansen's General Formulae for Perturbations.

BY G. W. HILL.

The last form in which HANSEN expressed the perturbations of the mean anomaly and equated radius vector is exhibited by the following equations:

$$n_0 z = n_0 t + c_0 + \int \left\{ \overline{W} + \frac{h_0}{h} \left(\frac{\nu}{1+\nu} \right)^2 \right\} n_0 dt \,,$$

$$\nu = C - \frac{1}{2} \int \left(\frac{d\overline{W}}{d\zeta} \right) dt \,,$$

(Equations 36 and 37, p. 97.)*

It will be perceived that the right-hand member of the first of these involves three quantities, viz. $\overline{W}$, ν and $\frac{h_0}{h}$. But the last of these quantities has no share in defining the position of the body, and it is desirable to get rid of it, provided that can be done without complicating the equation. This is readily accomplished by means of the equation (33, p. 95)

$$\frac{dz}{dt} = \frac{h_0}{h\,(1+\nu)^2} \,.$$

The result is

$$n_0 z = n_0 t + c_0 + \int \frac{\overline{W} + \nu^2}{1 - \nu^2} \, n_0 dt \,.$$

Why HANSEN has not put the equation in this form I cannot imagine; the advantage, not only as regards simplicity of expression, but also in point of ease of computation, is evident.

* See *Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten. Von P. A. Hansen. Erste Abhandlung.* (*Abhandlungen der Königlichen Sächsischen Gesellschaft der Wissenschaften. Band* III.) The numbering of the equations and the paging are from this volume.

HANSEN develops $\overline{W}$ by TAYLOR'S theorem, and, limiting ourselves to the second power of the disturbing force, we have

$$\overline{W} = \overline{W}_0 + \left(\frac{d\overline{W}_0}{d\gamma}\right) n_0\delta z = \overline{W}_0 - 2\,\frac{d\nu}{dt}\,\delta z\,.$$

When this value is substituted for $\overline{W}$ in the equation for $n_0 z$, we have a differential equation of the first order and degree for the determination of δz, the integral of which is well known. Terms of three dimensions with respect to disturbing forces being neglected, this procedure furnishes the equation

$$n_0\delta z = (1 - 2\nu)\int[\,(1 + 2\nu)\,\overline{W}_0 + \nu^2]\,n_0 dt\,,$$

which, however, is without interest other than analytical, as its use involves more labor than that of the equation given by HANSEN.

HANSEN'S equation for the determination of ν has the disadvantage of not affording the constant term of this quantity, and is inconvenient in computing the portion, of the form

$$At + Bt^2 + Ct^3 + \ldots\ldots,$$

which is independent of the arguments g, g', &c., as the values of A, B, &c., must be determined to a degree of accuracy much beyond what is necessary in the case of the other terms. As all the arbitrary constants admissible have been introduced by the integrations which give z, it is evident there must exist an equation determining ν without additional integrations. HANSEN has virtually employed this in the place where he shows how the constant term of ν is to be obtained, but has nowhere given it explicitly. This lacuna I propose to fill here.

The equation 39, p. 97,

$$W_0 = 2\frac{h}{h_0} - \frac{h_0}{h} - 1 + 2\frac{h}{h_0}\xi\,\frac{\rho}{a_0}\cos\omega + 2\frac{h}{h_0}\eta\,\frac{\rho}{a_0}\sin\omega\,,$$

may be employed to discover the value of $\frac{h}{h_0}$. The known expressions for $\frac{\rho}{a_0}\cos\omega$ and $\frac{\rho}{a_0}\sin\omega$ are

$$\frac{\rho}{a_0}\cos\omega = -\frac{3}{2}e + \left(J^{(0)}_{\frac{e}{2}} - J^{(2)}_{\frac{e}{2}}\right)\cos\gamma + \frac{1}{2}\left(J^{(1)}_{e} - J^{(3)}_{e}\right)\cos 2\gamma + \ldots,$$

$$\frac{\rho}{a_0}\sin\omega = \qquad \left(J^{(0)}_{\frac{e}{2}} + J^{(2)}_{\frac{e}{2}}\right)\sin\gamma + \frac{1}{2}\left(J^{(1)}_{e} + J^{(3)}_{e}\right)\sin 2\gamma + \ldots,$$

where HANSEN's notation for the BESSELIAN function is employed, and the subscript zero, which properly belongs to e, is, for convenience in writing, omitted. In his memoirs, where the mean anomaly is employed as the independent variable, HANSEN directs to compute only the parts of W_0 which are independent of γ or which have $\pm\gamma$ in their arguments; that is, the parts which have the form

$$X_0 + X_1 \cos\gamma + X_2 \sin\gamma,$$

X_0, X_1 and X_2 being independent of γ.

It will be easily perceived that, if we put

$$P = \frac{3}{2}\frac{e}{J^{(0)}_{\frac{e}{2}} - J^{(2)}_{\frac{e}{2}}},$$

P being thus a constant, the three first terms of W_0 must have the value

$$2\frac{h}{h_0} - \frac{h_0}{h} - 1 = X_0 + PX_1.$$

In this equation we may substitute for $\frac{h_0}{h}$ its value obtained from equation 33, and thus we obtain

$$\frac{2}{(1+\nu)^2\left(1+\frac{d\,.\,\delta z}{dt}\right)} - (1+\nu)^2\left(1+\frac{d\,.\,\delta z}{dt}\right) - 1 = X_0 + PX_1.$$

This equation, when $\frac{d\,.\,\delta z}{dt}$ is known, gives ν without additional integrations. To put it into a form suitable for computation, we add to each member such a quantity as will make the first equal to -6ν, then dividing both members by -6 we get

$$\nu = -\frac{1}{6}\left[X_0 + PX_1\right] - \frac{1}{2}\left[(1+\nu)^2\frac{d\,.\,\delta z}{dt} + \nu^2\right] + \frac{\left[(1+\nu)^2\frac{d\,.\,\delta z}{dt} + 2\nu + \nu^2\right]^2}{3(1+\nu)^2\left(1+\frac{d\,.\,\delta z}{dt}\right)}.$$

This equation is rigorous. If we may restrict ourselves to terms of the first order with respect to disturbing forces, it reduces to

$$\nu = -\frac{1}{6}\left[X_0 + PX_1\right] - \frac{1}{2}\frac{d\,.\,\delta z}{dt},$$

or, if terms of the second order must be included, to

$$\nu = -\frac{1}{6}\left[X_0 + PX_1\right] - \frac{1}{2}\frac{d\,.\,\delta z}{dt} + \frac{1}{3}\left[\frac{d\,.\,\delta z}{dt} + \frac{1}{2}\nu\right]^2 + \frac{3}{4}\nu^2.$$

The function usually tabulated is com. log $(1 + \nu)$; and we have com. log $(1 + \nu) = M\left\{-\frac{1}{6}\left[X_0 + PX_1\right] - \frac{1}{2}\frac{d\,.\,\delta z}{dt} + \frac{1}{3}\left[\frac{d\,.\,\delta z}{dt} + \frac{1}{2}\nu\right]^2 + \frac{1}{4}\nu^2\right\}$, M being the modulus of common logarithms.

These equations are as readily used as those given by HANSEN, and are free from the disadvantages, previously mentioned, which belong to the latter. All the quantities involved, except X_1, have already been obtained in the computation of δz. Also X_1 is readily got by putting $\gamma = 0$ in the terms of W_0 which involve this quantity, and summing two and two together the terms which result.

WASHINGTON, D. C., *December* 17, 1881.

On the Solution of a Certain Class of Difference or Differential Equations.

By J. J. Sylvester.

Casting my eye over Mr. Moulton's valuable edition of Boole's Treatise on Finite Differences (see pp. 229–231), I was gratified to find that he had embalmed in it a solution that I had given many years ago, of an equation in differences, of the simple but very general form expressed by equating to zero or to Pm^x the persymmetrical determinant

$$\begin{vmatrix} u_x & u_{x+1} & \cdots & u_{x+i} \\ u_{x+1} & u_{x+2} & \cdots & u_{x+i+1} \\ u_{x+2} & u_{x+3} & \cdots & u_{x+i+2} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_{x+i} & u_{x+i+1} & \cdots & u_{x+2i} \end{vmatrix}$$

which is of the i^{th} degree and $2i^{\text{th}}$ order.

To fix the ideas, let us consider the simple case

$$u_x u_{x+2} - u_{x+1}^2 = Pm^x,$$

of which, when $P = 0$, the solution is $u_x = A\alpha^x$, A and α being both arbitrary, but for P not zero is expressed by $u_x = \pm (A\alpha^x + B\beta^x)$ with the conditions

$$\alpha\beta = m, \quad AB(\alpha - \beta)^2 = P$$

which solution as an *obiter dictum* I may remark may easily be converted into the simpler and more explicit form

$$(\sin\beta)^2 u_x^2 + P(\sin\overline{\alpha + \beta x})^2 m^{x-1} = 0$$

where α, β are arbitrary constants.

If we proceed now to verify the solution in its original form, we shall immediately be led to perceive a certain generalization which the given equation may be made to undergo without ceasing to be soluble—the solution however becoming narrowed from a general to a special one: whether particular or singular I shall not discuss.

If we write $u_x = A\alpha^x + B\beta^x$, the determinant becomes

$$\begin{vmatrix} A\alpha^x + B\beta^x & A\alpha^{x+1} + B\beta^{x+1} \\ A\alpha^{x+1} + B\beta^{x+1} & A\alpha^{x+2} + B\beta^{x+2} \end{vmatrix}$$

which is equal to $AB(\alpha-\beta)^2(\alpha\beta)^x$; this is the verification spoken of: but, as a consequence, it is apparent that we must have

$$\begin{vmatrix} A\alpha^x + B\beta^x + C\gamma^x, & A\alpha^{x+1} + B\beta^{x+1} + C\gamma^{x+1} \\ A\alpha^{x+1} + B\beta^{x+1} + C\gamma^{x+1} & A\alpha^{x+2} + B\beta^{x+2} + C\gamma^{x+2} \end{vmatrix}$$
$$= AB(\alpha-\beta)^2(\alpha\beta)^x + BC(\beta-\gamma)^2(\beta\gamma)^x + CA(\gamma-\alpha)^2(\gamma\alpha)^x.$$

Hence we can solve the equation

$$u_x u_{x+2} - u_{x+1}^2 = Pl^x + Qm^x + Rn^x,$$

viz. we may write

$$u_x + A\alpha^x + B\beta^x + C\gamma^x = 0,$$

where

$$\beta\gamma = l, \quad \gamma\alpha = m, \quad \alpha\beta = n,$$

$$AB(\alpha-\beta)^2 = R, \quad BC(\beta-\gamma)^2 = P, \quad CA(\gamma-\alpha)^2 = Q,$$

that is to say

$$\alpha = \sqrt{\frac{mn}{l}}, \quad \beta = \sqrt{\frac{nl}{m}}, \quad \gamma = \sqrt{\frac{lm}{n}},$$

$$A = \sqrt{\frac{QR}{P}}\,\frac{(\beta-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}, \quad B = \sqrt{\frac{RP}{Q}}\,\frac{(\gamma-\alpha)}{(\beta-\alpha)(\beta-\gamma)}, \quad C = \sqrt{\frac{PQ}{R}}\,\frac{(\alpha-\beta)}{(\gamma-\alpha)(\gamma-\beta)}.$$

or calling

$$\sqrt{lmn} = g, \quad \sqrt{PQR} = G,$$

$$\alpha = \frac{g}{l}, \quad \beta = \frac{g}{m}, \quad \gamma = \frac{g}{n},$$

$$A = \frac{G}{g}\cdot\frac{(n-m)\,l^2}{(l-m)(l-n)P}, \quad B = \frac{G}{g}\cdot\frac{(l-n)\,m^2}{(m-n)(m-l)\,Q}, \quad C = \frac{G}{g}\cdot\frac{(m-l)\,n^2}{(n-l)(n-m)\,R}.$$

The result therefore in its rational unambiguous form is

$$PQR\,(lm-mn)^2\,(mn-nl)^2\,(nl-lm)^2\,(lmn)^{2x-1}\,u_x^2=\{\Sigma\,(lm-ln)\,QR\,(mn)^x\}^2.$$

When any of the quantities P, Q, R vanish, or any of the quantities l, m, n vanish or become equal to one another, the solution fails.

We shall, however, easily obtain a compensatory form of equation supplying the place of two of the exponentials, and another supplying the place of all three becoming identical, and the solution of these substituted forms may be deduced from that of the original form of the equation.

Thus, first, let

$$\left.\begin{array}{ll} m=(1+\varepsilon)\,\mu & n=(1-\varepsilon)\,\mu \\ Q=\frac{1}{2}\left(S+\frac{T}{\varepsilon}\right) & R=\frac{1}{2}\left(S-\frac{T}{\varepsilon}\right) \end{array}\right\} \text{ where } \varepsilon \text{ is an infinitesimal.}$$

Then the equation becomes

$$u_x\,u_{x+2}-u_{x+1}^2=Pl^x+S\mu^x+Tx\mu^x$$

and the solution in its unreduced form is

$$u_x=A\alpha^x+B\beta^x+C\gamma^x,$$

where

$$\alpha=\sqrt{\frac{\mu^2}{l}},\quad \beta=(1-\varepsilon)\sqrt{l},\quad \gamma=(1+\varepsilon)\sqrt{l},$$

and

$$A=\sqrt{\frac{PR}{Q}}\;\frac{\beta-\gamma}{(\alpha-\beta)\,(\alpha-\gamma)}=\frac{T}{\sqrt{-P}}\;\frac{\sqrt{l}}{\left(\sqrt{\frac{\mu^2}{l}}-\sqrt{l}\right)^2}=\frac{T}{\sqrt{-P}}\cdot\frac{l^{\frac{3}{2}}}{(\mu-l)^2}$$

$$B=\sqrt{\frac{PR}{Q}}\,\frac{\gamma-\alpha}{(\beta-\alpha)\,(\beta-\gamma)}=\sqrt{-P}\left(1-2\,\varepsilon\frac{S}{T}\right)\frac{\sqrt{l}-\sqrt{\frac{\mu^2}{l}}+\sqrt{l}\,\varepsilon}{\sqrt{l}-\sqrt{\frac{\mu^2}{l}}-\sqrt{l}\,\varepsilon}\div(-2\,\varepsilon\sqrt{l})$$

$$=\sqrt{-P}\,\left\{\frac{-1}{2\sqrt{l}\,\varepsilon}+\frac{S}{T\sqrt{l}}-\frac{\sqrt{l}}{l-\mu}\right\}$$

$$C=\sqrt{-P}\,\left\{\frac{1}{2\sqrt{l}\,\varepsilon}+\frac{S}{T\sqrt{l}}-\frac{\sqrt{l}}{l-\mu}\right\}$$

Hence $$B\beta^x + C\gamma^x = \sqrt{-P}\, l^{\frac{x-1}{2}} \left\{ \begin{array}{l} (1+x\varepsilon)\left(\frac{1}{2\varepsilon} + \frac{S}{T} - \frac{l}{l-\mu}\right) \\ +(1-x\varepsilon)\left(-\frac{1}{2\varepsilon} + \frac{S}{T} - \frac{l}{l-\mu}\right) \end{array} \right\}$$

$$= \sqrt{-P}\left\{2\left(\frac{S}{T} - \frac{l}{l-\mu}\right) + x\right\} l^{\frac{x-1}{2}};$$

so that

$$\sqrt{-lP}\, u_x = T\left(\frac{l}{\mu-l}\right)^2\left(\frac{\mu^2}{l}\right)^{\frac{x}{2}} - P\left(\frac{2S}{T} - \frac{2l}{l-\mu} + x\right) l^{\frac{x}{2}}$$

or

$$Pl^{x+1} u_x^2 + \left\{T\left(\frac{l}{\mu-l}\right)^2 \mu^x - P\left(\frac{2S}{T} + \frac{2l}{\mu-l} + x\right) l^x\right\}^2 = 0$$

will satisfy the given equation

$$u_x\, u_{x+2} - u_{x+1}^2 = Pl^x + S\mu^x + Tx\mu^x.$$

When $T = 0$ the solution fails, as we know *à priori* it ought to do.

When $S = 0$ it takes the form

$$Pl^{x+1} u_x^2 + \left\{T\left(\frac{l}{\mu-l}\right)^2 \mu^x - P\left(\frac{2l}{\mu-l} + x\right) l^x\right\}^2 = 0.$$

We might, by an analogous process, writing $(1+\varepsilon)$, $(1+\rho\,\varepsilon)$, $(1+\rho^2\varepsilon)$ in lieu of l, m, n, and giving P, Q, R appropriate values involving ε^2 as well as ε, render $\Sigma P l^x$ a finite function of the form $(S + Tx + Ux^2)\lambda^x$, and deduce the solution of $u_x\, u_{x+2} - u_{x+1}^2 = (S + Tx + Ux^2)\lambda^x$ as a particular case of the solution of the general equation. But as we can easily see that the unreduced form of the solution must be $u_x = \lambda^{\frac{x}{2}}(A + Bx + Cx^2)$, it will be easier to find A, B, C immediately from the equation

$$\begin{vmatrix} A + Bx + Cx^2 & A + B(x+1) + C(x+1)^2 \\ A + B(x+1) + C(x+1)^2 & A + B(x+2) + C(x+2)^2 \end{vmatrix}$$

or $$\begin{vmatrix} A + Bx + Cx^2 & A + B(x+1) + C(x+1)^2 \\ B + C + 2x & B + 3C + 2Cx \end{vmatrix}$$

or $$\begin{vmatrix} A + Bx + Cx^2 & B + C + 2Cx \\ B + C + 2Cx & 2C \end{vmatrix} = S + Tx + Ux^2.$$

Hence $\quad -2C^2 = U, \; -2BC - 4C^2 = T, \; 2AC - (B+C)^2 = S.$

Hence

$$C=\sqrt{\frac{-U}{2}},\quad B=-2C-\frac{T}{2C}=-\sqrt{-2U}-\frac{T}{\sqrt{-2U}}=\frac{2U+T}{\sqrt{-2U}},$$

$$A=\frac{S+(B+C)^2}{2C}=\frac{S+\left(\sqrt{\frac{-U}{2}}+\frac{T}{\sqrt{-2U}}\right)^2}{\sqrt{-2U}}=\frac{-2SU+(T+U)^2}{-2U\sqrt{-2U}},$$

or

$$8U^3u_x^2+\{2U^2x^2+(4U^2-2UT)x+2SU-(T+U)^2\}\lambda^x=0$$

is the required primitive of the given equation.

The method may obviously be extended to any equation of the given form: that is to say when the persymmetrical determinant which it contains is of the degree i and is equated to $(i+1)$ multiples of exponentials each of the form Pl^x an integral of it can be found, and if these i exponentials be subdivided into partial groups of ε, ε', ε'' . . . terms in a group, then instead of the ε multiples of exponentials belonging to any group may be substituted $(P_1+P_2x+P_3x^2+\ldots+P_\varepsilon x^{\varepsilon-1})l^x$, and the solution of the equation so modified may be deduced from the solution first mentioned as a particular case thereof.

It will be sufficient for all reasonable purposes of illustration briefly to consider the case of

$$\begin{vmatrix} u_x & u_{x+1} & u_{x+2} \\ u_{x+1} & u_{x+2} & u_{x+3} \\ u_{x+2} & u_{x+3} & u_{x+4} \end{vmatrix}=Pl^x+Qm^x+Rn^x+Sp^x.$$

An integral of this may be found by writing

$$u_x=A\alpha^x+B\beta^x+C\gamma^x+D\delta^x,$$

where

$$\beta\gamma\delta=l,\quad \alpha\gamma\delta=m,\quad \alpha\beta\delta=n,\quad \alpha\beta\gamma=p,$$

$$BCD\zeta(\beta,\gamma,\delta)=P,\qquad ACD\zeta(\alpha,\gamma,\delta)=Q,\qquad ABD\zeta(\alpha,\beta,\delta)=R,$$

$$ABC\zeta(\alpha,\beta,\gamma)=S,$$

ζ meaning the product of the squared differences of the letters which it governs.

We have thus

$$\alpha=\frac{g}{l},\quad \beta=\frac{g}{m},\quad \gamma=\frac{g}{n},\quad \delta=\frac{g}{p},$$

where

$$g=\sqrt[3]{lmnp}$$

and

$$A^3B^3C^3D^3[\zeta(\alpha,\beta,\gamma,\delta)]^2=PQRS,$$

so that writing

$$G = \left\{ \frac{PQRS}{[\zeta(\alpha, \beta, \gamma, \delta)]^2} \right\}^{\frac{1}{3}} = \frac{1}{g^3} \frac{(PQRS)^{\frac{1}{3}}}{\left\{ \zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right) \right\}^{\frac{2}{3}}}$$

$$A = \zeta(\beta, \gamma, \delta) \frac{G}{P};\ B = \zeta(\alpha, \gamma, \delta) \frac{G}{Q};\ C = \zeta(\alpha, \beta, \delta) \frac{G}{R};\ D = \zeta(\alpha, \beta, \gamma) \frac{G}{S};$$

and thus

$$(PQRS)^2 (lmnp)^3 \left\{ \zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right) \right\}^2 u_x^3 = (lmnp)^x \{\Sigma\, QRS\zeta(\beta, \gamma, \delta) l^{-x}\}^3$$

or

$$\left\{ PQRS\, \zeta\left(\frac{1}{l}, \frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right) \right\}^2 u_x^3 = (lmnp)^{x-3} \left\{ \Sigma\, QRS\zeta\left(\frac{1}{m}, \frac{1}{n}, \frac{1}{p}\right) l^{-x} \right\}^3.$$

It is scarcely necessary to add that all the above conclusions continue to hold, when, on the left hand side of the equation for u_{x+h} we write $\left(\frac{d}{dx}\right)^h y$ and at the same time for any exponential l^x on the right hand side substitute e^{lx}.

Thus for instance we may in general find an integral of

$$yy'' - y'^2 = Ae^{hx} + Be^{kx} \cos(\alpha x + \beta)$$

or again of

$$(yy'' - y'^2)y'''' - y(y''')^2 + 2y'y''y''' - y''^3 = Ae^{hx} \cos(\alpha x + \beta) + Be^{kx} \cos(\gamma x + \delta).$$

On the Analytical Forms called Trees.

By Professor Cayley.

In a tree of N knots, selecting any knot at pleasure as a root, the tree may be regarded as springing from this root, and it is then called a root-tree. The same tree thus presents itself in various forms as a root-tree; and if we consider the different root-trees with N knots, these are not all of them distinct trees. We have thus the two questions, to find the number of root-trees with N knots; and, to find the number of distinct trees with N knots.

I have in my paper "On the Theory of the Analytical Forms called Trees," *Phil. Mag.*, t. 13 (1857), pp. 172–176, given the solution of the first question; viz. if ϕ_N denotes the number of the root-trees with N knots, then the successive numbers ϕ_1, ϕ_2, ϕ_3, etc., are given by the formula

$$\phi_1 + x\phi_2 + x^2\phi_3 + \ . \ . \ . = (1-x)^{-\phi_1}(1-x^2)^{-\phi_2}(1-x^3)^{-\phi_3} \ . \ . \ .$$

viz. we thus find

suffix of ϕ	1	2	3	4	5	6	7	8	9	10	11	12	13
$\phi =$	1	1	2	4	9	20	48	115	286	719	1842	4766	12486

And I have, in the paper "On the Analytical Forms called Trees, with Applications to the Theory of Chemical Combinations," *Brit. Assoc. Report*, 1875, pp. 257–305, also shown how by the consideration of the centre or bicentre "of length" we can obtain formulae for the number of central and bicentral trees, that is for the number of distinct trees, with N knots: the numerical result obtained for the total number of distinct trees with N knots is given as follows:

No. of Knots	1	2	3	4	5	6	7	8	9	10	11	12	13
No. of Central Trees	1	0	1	1	2	3	7	12	27	55	127	284	682
" Bicentral "	0	1	0	1	1	3	4	11	20	51	108	267	619
Total	1	1	1	2	3	6	11	23	47	106	235	551	1301

But a more simple solution is obtained by the consideration of the centre or bicentre "of number." A tree of an odd number N of knots has a centre of number, and a tree of an even number N of knots has a centre or else a bicentre of number. To explain this notion (due to M. Camille Jordan) we consider the branches which proceed from any knot, and (excluding always this knot itself) we count the number of the knots upon the several branches; say these numbers are α, β, γ, δ, ε, etc., where of course $\alpha+\beta+\gamma+\delta+\varepsilon+\text{etc.}=N-1$. If N is even we may have, say $\alpha=\frac{1}{2}N$; and then $\beta+\gamma+\delta+\varepsilon+\text{etc.}=\frac{1}{2}N-1$, viz. α is larger by unity than the sum of the remaining numbers: the branch with α knots, or the number α, is said to be "merely dominant." If N be odd, we cannot of course have $\alpha=\frac{1}{2}N$, but we may have $\alpha>\frac{1}{2}N$; here α exceeds by 2 at least the sum of the other numbers; and the branch with α knots, or the number α, is said to be "predominant." In every other case, viz. in the case where each number α is less than $\frac{1}{2}N$, (and where consequently the largest number α does not exceed the sum of the remaining numbers), the several branches, or the numbers α, β, γ, etc., are said to be subequal. And we have the theorem: first when N is odd, there is always one knot (and only one knot) for which the branches are subequal: such knot is called the centre of number. Secondly when N is even; either there is one knot (and only one knot) for which the branches are subequal; and such knot is then called the centre of number; or else there is no such knot, but there are two adjacent knots (and no other knot) each having a merely-dominant branch; such two knots are called the bicentre of number, and each of them separately is a half-centre.

Considering now the trees with N knots as springing from a centre or a bicentre of number, and writing ψ_N for the whole number of distinct trees with N knots, we readily obtain these in terms of the foregoing numbers ϕ_1, ϕ_2, ϕ_3, etc., viz. we have

$$\begin{aligned}
\psi_1 &= 1,\\
\psi_2 &= \tfrac{1}{2}\phi_1(\phi_1+1),\\
\psi_3 &= \text{coeff. } x^2 \text{ in } (1-x)^{-\phi_1},\\
\psi_4 &= \tfrac{1}{2}\phi_2(\phi_2+1) + \text{coeff. } x^3 \text{ in } (1-x)^{-\phi_1},\\
\psi_5 &= \text{coeff. } x^4 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2},\\
\psi_6 &= \tfrac{1}{2}\phi_3(\phi_3+1) + \text{coeff. } x^5 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2},\\
\psi_7 &= \text{coeff. } x^6 \text{ in } (1-x)^{-\phi_1}(1-x^2)^{-\phi_2}(1-x^3)^{-\phi_3},
\end{aligned}$$

and so on, the law being obvious. And the formulae are at once seen to be true. Thus for $N=6$, the formula is

$$\psi_6 = \tfrac{1}{2}\phi_3(\phi_3+1) + \tfrac{1}{2}\phi_2(\phi_2+1).\phi_1 + \phi_2.\tfrac{1}{6}\phi_1(\phi_1+1)(\phi_1+2)$$
$$+ \tfrac{1}{120}\phi_1(\phi_1+1)(\phi_1+2)(\phi_1+3)(\phi_1+4).$$

We have ϕ_3 root-trees with 3 knots, and by simply joining together any two of them, treating the two roots as a bicentre, we have all the bicentral trees with 6 knots: this accounts for the term $\tfrac{1}{2}\phi_3(\phi_3+1)$. Again we have ϕ_1 root-trees with 1 knot, ϕ_2 root-trees with 2 knots; and with a given knot as centre, and the partitions (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1) successively, we build up the central trees of 6 knots, viz. 1° we take as branches any two ϕ_2's and any one ϕ_1; 2° any one ϕ_2 and any three ϕ_1's; 3° any five ϕ_1's; the partitions in question being all the partitions of 5 with no part greater than 2, that is all the partitions with subequal parts. We easily obtain

suffix of ψ	1	2	3	4	5	6	7	8	9	10	11	12	13
$\psi=$	1	1	1	2	3	6	11	23	47	106	235	551	1301

agreeing with the results obtained by the much more complicated formulae of the paper of 1875.

Notes.

I.

On Symbols of Operation.

By Professor Crofton, F. R. S.

To prove that, ϕ being any function of D, *i. e.* $\frac{d}{dx}$,

$$e^{x\phi(D)}e^{hx}=e^{\lambda x}, \tag{1}$$

where λ is a constant determined from

$$\psi(\lambda)=1+\psi(h), \tag{2}$$

the function ψ being defined by

$$\psi(x)=\int\frac{dx}{\varphi(x)}. \tag{3}$$

The above theorem might be made to follow from principles given in a paper by me in the Proceedings of the London Mathematical Society, April 1881; but the following method may be also employed.

Let $$u=e^{kx\phi(D)}e^{hx};$$

differentiating with regard to h,

$$\frac{du}{dh}=e^{kx\phi D}xe^{hx},$$

also $$\frac{du}{dk}=e^{kx\phi D}x\phi(D)e^{hx}$$
$$=\phi(h)e^{kx\phi D}xe^{hx}.$$

We have thus a partial differential equation

$$\frac{du}{dk}=\phi(h)\frac{du}{dh}$$

$$\therefore\quad u=\chi\left(k+\int\frac{dh}{\varphi(h)}\right)=\chi\left(k+\psi(h)\right).$$

To determine the arbitrary function χ, we observe that if $k=0$, $u=e^{hx}$;

$$\therefore\quad \chi(\psi h)\equiv e^{hx}$$

hence if λ be determined so as to satisfy

$$\psi(\lambda)\equiv k+\psi(h)$$
$$u=\chi(\psi\lambda)=e^{\lambda x}.$$

For instance, let $\phi(D) = D^2$; $\psi(x) = -\frac{1}{x}$ by (3); hence by (2), $\lambda = \frac{h}{1-h}$;

$$\therefore \quad e^{xD^2} e^{hx} = e^{\frac{hx}{1-h}}$$

Again $$e^{kxD^{-1}} e^{hx} = e^{x\sqrt{h^2+2k}}.$$

Again $$e^{kxD^r} e^{hx} = e^{\lambda x},$$

where $$\lambda = [h^{1-r} + (1-r)k]^{\frac{1}{1-r}}.$$

The above process may sometimes be applied in other similar cases; for example, to find

$$u = e^{kx^{-1}D^2} e^{hx^3}:$$

we may deduce the equation

$$\frac{du}{dk} = 9h^2 \frac{du}{dh} + 6hu;$$

the solution of this by Lagrange's method, or otherwise, is

$$u = h^{-\frac{2}{3}} \phi(h^{-1} - 9k);$$

and determining ϕ from the condition that $u = e^{hx}$ when $k = 0$, we find

$$u = (1 - 9hk)^{-\frac{1}{3}} \exp.\left(\frac{hx^3}{1 - 9hk}\right).$$

A very remarkable identity between certain symbolic operators of the exponential form may be established by a comparison of Lagrange's theorem and a result which I have given in the paper above referred to. It is there shown that if z is related to x by the equation

$$\psi(z) = 1 + \psi(x) \quad \ldots \quad (1)$$

and if we put for shortness

$$\frac{1}{\psi'(x)} = \phi(x) = \phi \quad \ldots \quad (2)$$

then $$e^{\phi D} F(x) = F(z) \quad \ldots \quad (3)$$

also $$e^{D\phi} F(x) = F(z) \frac{dz}{dx} \quad \ldots \quad (4)$$

whatever be the function F.

Now if we use for shortness the symbol $e^{D \cdot f(x)}$ or $e^{D \cdot f}$ to express the development

$$e^{D \cdot f} \equiv 1 + Dfx + \frac{1}{1.2} D^2 (fx)^2 + \&c., \quad \ldots \quad (5)$$

Lagrange's theorem gives

$$e^{D \cdot f} F(x) \equiv F(z) \frac{dz}{dx} \quad \ldots \quad (6)$$

where $$z = x + f(z) \quad \ldots \quad (7)$$

Hence if z be the same function of x in (6) as in (4), the operator $e^{D\phi}$ is identical with $e^{D\cdot f}$; that is

$$1 + D\phi + \frac{1}{1.2}\overline{D\phi}|^2 + \&c. \equiv 1 + Df + \frac{1}{1.2}D^2f^2 + \&c. \quad . \; . \; . \; . \; (8)$$

In order that this identity should hold, it is necessary that z, $f(x)$, $\psi(x)$ shall be three functions of x which satisfy (1) and (7), viz.

$$z = x + f(z) \; . \; . \; . \; . \; (7)$$
$$\psi(z) = 1 + \psi(x) \; . \; . \; . \; . \; (1)$$

$\phi(x)$ being put for $\frac{1}{\psi'(x)}$.

If we could then eliminate (z) from (1) and (7), any functions f, ψ, whose forms satisfy the resulting equation, will cause the identity (8) to hold.

For instance, suppose $\phi(x) = hx^2$, then $\psi x = -\frac{1}{hx}$;

hence (1)

$$x = \frac{z}{1 + hz}$$

therefore (7) $\qquad f(z) \equiv \frac{hz^2}{1 + hz}$

therefore $1 + hDx^2 + \frac{h^2}{1.2}\overline{Dx^2}|^2 + \&c. \equiv 1 + hD\frac{x^2}{1 + hx} + \frac{h^2}{1.2}D^2\left(\frac{x^2}{1 + hx}\right)^2 + \&c.$

whatever be the operand.

Again let us suppose $f(x) = hx$, then $z = \frac{x}{1 - h}$;

hence (7) ψ is to be found from

$$\psi\left(\frac{x}{1 - h}\right) \equiv 1 + \psi(x)$$

$$\therefore \quad \psi(x) \equiv -\frac{\log x}{\log(1 - h)}$$

$$\therefore \quad \phi(x) = -x\log(1 - h)$$

Thus

$$1 + hDx + \frac{h^2}{1.2}D^2x^2 + \; . \; . \; . \; . \equiv e^{-\log(1-h)Dx} \equiv (1 - h)^{-Dx}$$

$$\equiv 1 + hDx + \frac{h^2}{1.2}Dx(Dx + 1) + \frac{h^3}{1.2.3}Dx(Dx + 1)(Dx + 2) + \; . \; . \; . \; .$$

This is easy to verify.

II.

On Segments made on Lines by Curves.

By Miss Christine Ladd, *Johns Hopkins University.*

The two theorems in regard to the segments made on a line by a fixed point and the intersections of the line with a curve, which are given in Salmon's *Higher Plane Curves*, Art. 123, are special cases of a more general theorem, which may be expressed in this way:

Through each of two fixed points, M, N, a variable line, μ, ν, is drawn. These lines intersect in the variable point X, and cut a curve of the n^{th} order in the points $m_1, m_2, \ldots m_n$ and $n_1, n_2, \ldots n_n$ respectively. Then the ratio

$$\frac{Mm_1 \,.\, Mm_2 \ldots\ldots}{Nn_1 \,.\, Nn_2 \ldots\ldots} : \frac{Xm_1 \,.\, Xm_2 \ldots\ldots}{Xn_1 \,.\, Xn_2 \ldots\ldots}$$

is constant.

For if A is the constant term in the equation of the curve, and if we write $F(\theta)$ for the coefficient of the term of highest degree in the equation transformed to polar coördinates, θ_μ for the angle which the line μ makes with the axis of x, &c., A_M for the value of the constant term when the point M is made the origin, &c., then we have

$$Mm_1 \,.\, Mm_2 \ldots\ldots = A_M : F(\theta_\mu)$$
$$Xn_1 \,.\, Xn_2 \ldots\ldots = A_X : F(\theta_\nu), \text{ \&c.}$$

Hence

$$\frac{Mm_1 \,.\, Mm_2 \ldots}{Nn_1 \,.\, Nn_2 \ldots\ldots} \text{ into } \frac{Xn_1 \,.\, Xn_2 \ldots\ldots}{Xm_1 \,.\, Xm_2 \ldots\ldots} \qquad (p)$$

$$= \frac{A_M}{F(\theta_\mu)} \cdot \frac{F(\theta_\nu)}{A_N} \text{ into } \frac{A_X}{F(\theta_\nu)} \cdot \frac{F(\theta_\mu)}{A_X} = A_M : A_N,$$

which is constant, since the coördinates of M and N are constant.

When the points M and N are at infinity the first factor in (p) becomes equal to unity, and the theorem shows that the ratio of the products of segments on two lines of fixed direction through a variable point X is constant. When the variable point X is at infinity the second factor in (p) becomes equal to unity and we see that the ratio of the products of segments on two parallel lines of variable direction through two fixed points is constant. It happens that the latter ratio has the same value as the compound ratio (p).

III.

On the Multiplication of the $(n-1)^{th}$ Power of a Symmetric Determinant of the n^{th} Order by the Second Power of any Determinant of the same Order.

BY THOMAS MUIR, M. A., F. R. S. E.

1. In *Crelle's Journal*, 1853, Vol. XLIX, p. 246, § 3, Hesse gives the identity

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}^2 = \begin{Vmatrix} \begin{vmatrix} a & b & x_1 \\ b & c & x_2 \\ x_1 & x_2 & 0 \end{vmatrix} & \begin{vmatrix} a & b & x_1 \\ b & c & x_2 \\ y_1 & y_2 & 0 \end{vmatrix} \\ \begin{vmatrix} a & b & x_1 \\ b & c & x_2 \\ y_1 & y_2 & 0 \end{vmatrix} & \begin{vmatrix} a & b & y_1 \\ b & c & y_2 \\ y_1 & y_2 & 0 \end{vmatrix} \end{Vmatrix}$$

adding that in a subsequent part of his paper he would show of what extension the identity was capable. This he did in § 6, p. 252. Starting the process of generalisation with his mind fixed on the left-hand member, he reached the proposition that "*The product of a symmetrical determinant by the second power of any determinant of the same order is expressible as a symmetrical determinant.*" The object of the present note is to show that for the same identity there exists an extension in quite another direction, the starting point now being the *right-hand* member of it.

2. Taking the symmetric determinant of the third order

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}$$

and bordering it with pairs of the rows of the determinant $| x_1\, y_2\, z_3 |$ after the manner indicated in (1) we obtain six determinants; and making these the six distinct elements of a symmetric determinant of the third order, we have an

expression perfectly similar to the right-hand member of Hesse's identity. The condensation of this symmetric determinant leads to the following theorem:

$$\left|\begin{matrix} \begin{vmatrix} a & b & c & x_1 \\ b & d & e & x_2 \\ c & e & f & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & x_1 \\ b & d & e & x_2 \\ c & e & f & x_3 \\ y_1 & y_2 & y_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & x_1 \\ b & d & e & x_2 \\ c & e & f & x_3 \\ z_1 & z_2 & z_3 & 0 \end{vmatrix} \\ \begin{vmatrix} a & b & c & y_1 \\ b & d & e & y_2 \\ c & e & f & y_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & y_1 \\ b & d & e & y_2 \\ c & e & f & y_3 \\ y_1 & y_2 & y_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & y_1 \\ b & d & e & y_2 \\ c & e & f & y_3 \\ z_1 & z_2 & z_3 & 0 \end{vmatrix} \\ \begin{vmatrix} a & b & c & z_1 \\ b & d & e & z_2 \\ c & e & f & z_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & z_1 \\ b & d & e & z_2 \\ c & e & f & z_3 \\ y_1 & y_2 & y_3 & 0 \end{vmatrix} & \begin{vmatrix} a & b & c & z_1 \\ b & d & e & z_2 \\ c & e & f & z_3 \\ z_1 & z_2 & z_3 & 0 \end{vmatrix} \end{matrix}\right| = -\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix}^2 \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^2.$$

The proof is exceedingly simple. From one of Sylvester's theorems regarding Compound Determinants (*Philos. Mag.*, 1851, I, p. 297) we have

$$\left|\begin{matrix} |a_1\, b_2\, c_3\, d_4| & |a_1\, b_2\, c_3\, e_4| & |a_1\, b_2\, c_3\, f_4| \\ |a_1\, b_2\, c_3\, d_5| & |a_1\, b_2\, c_3\, e_5| & |a_1\, b_2\, c_3\, f_5| \\ |a_1\, b_2\, c_3\, d_6| & |a_1\, b_2\, c_3\, e_6| & |a_1\, b_2\, c_3\, f_6| \end{matrix}\right| = |a_1\, b_2\, c_3\, d_4\, e_5\, f_6|\; |a_1\, b_2\, c_3|^2.$$

Replacing in this the determinant $|a_1\, b_2\, c_3\, d_4\, e_5\, f_6|$ by

$$\begin{vmatrix} a & b & c & x_1 & y_1 & z_1 \\ b & d & e & x_2 & y_2 & z_2 \\ c & e & f & x_3 & y_3 & z_3 \\ x_1 & x_2 & x_3 & 0 & 0 & 0 \\ y_1 & y_2 & y_3 & 0 & 0 & 0 \\ z_1 & z_2 & z_3 & 0 & 0 & 0 \end{vmatrix},$$

which is equal to $-|x_1\, y_2\, z_3|^2$, we at once obtain all that is required.

The form of the theorem for higher orders is apparent, and is indicated in the title of the note.

3. It is worthy of remark that Hesse's general theorem enunciated in (1) leads directly to another of at least equal importance, viz. *Any power of a symmetrical determinant is itself expressible as a symmetrical determinant.* For, the *first* power being symmetrical, the *third* must be so also; therefore also the *fifth*, the *seventh*, &c.; and as for the *even* powers, they are known to be symmetrical in the case of any determinant. In the order of discovery, however, this theorem preceded Hesse's, being used by Sylvester in the statement of the property found by him to belong to the equation of the secular inequalities of the planets. In this connection no proof was given of it except in the *Nouvelles Annales*, the editor of which, in bringing the property referred to before his readers, prefaced the statement of the same by a few remarks on determinants. To him apparently the responsibility attaches of discovering the theorem "Le produit de déterminants symétriques est un déterminant symétrique," in order that the accustomed easy step might be made from *product* to *power*.

BEECHCROFT, BISHOPTON, SCOTLAND, *August* 1st, 1881.

IV.

On Newton's Method of Approximation.

BY F. FRANKLIN.

Let $f(x) = 0$ be an algebraic equation. Newton's method of approximation consists in adding to an approximate value, a, of a root, the correction k, $= -\dfrac{f(a)}{f'(a)}$; correcting the new value $a + k$, say a_1, in like manner, viz. by adding $-\dfrac{f(a_1)}{f'(a_1)}$; and so on. Fourier's theorem concerning this method is as follows: If between a and b there is one and only one root of the equation $f(x) = 0$, and *if neither $f'(x)$ nor $f''(x)$ vanishes* between these limits, then we will be sure to approximate indefinitely to the root by Newton's method, if we begin the process at that one of the quantities a, b for which f has the same sign as f''.

The proof, as usually given, is somewhat tedious; the following proof is very brief, shows that a part of the usual statement of the theorem should be omitted, and gives immediately a measure of the rapidity of the approximation.

The Newtonian correction, k, is $-\dfrac{f(a)}{f'(a)}$. Denote the true correction by h, so that $a+h$ is the root. Then

$$0 = f(a+h) = f(a) + hf'(a) + \frac{1}{2}h^2 f''(a+\theta h),$$

whence

$$h = -\frac{f(a)}{f'(a)} - \frac{1}{2}h^2 \frac{f''(a+\theta h)}{f'(a)},$$

so that k has the same sign as h and is less in absolute value than h, provided $f''(a+\theta h)$ has the same sign as $f(a)$. That is, the corrected value $a+k$ (say a_1) will be nearer to the root than a and on the same side of the root as a, provided that f'' has, throughout the interval in question, the sign of $f(a)$; and since, if this condition is fulfilled, $f(a)$ and $f(a_1)$ have like signs, the same condition will be fulfilled for a_1. Thus the theorem above stated is proved, with the substitution, for the words in italics, of the words *if $f''(x)$ does not change sign.* That is, only one condition is required, the one relating to the *first* derived function being superfluous: the geometrical meaning of this fact is obvious.

The error after the first correction is

$$-\frac{1}{2}h^2 \frac{f''(a+\theta h)}{f'(a)},$$

i. e. it cannot exceed in absolute value the product of half the square of the original error by $\dfrac{F''}{f'(a)}$, where F'' denotes the numerically greatest value of $f''(x)$ between $x=a$ and $x=b$.

Serret, in his *Cours d'Algèbre Supérieure* (vol. 2, pp. 346–348), deduces by a long and somewhat difficult method the result that the error after the first approximation cannot exceed the product of $\frac{1}{2}h^2$ by the numerically greatest value of the fraction $\dfrac{f''(x)}{f'(x)}$ between $x=a$ and $x=b$. This is obviously less accurate than the result obtained above; and in fact it may be noted that $f'(a)$ is necessarily the numerically *greatest* value which the denominator, $f'(x)$, can take between the limits.

BALTIMORE, *May*, 1881.

Simple and Uniform Method of Obtaining Taylor's, Cayley's, and Lagrange's Series.

By J. C. Glashan, *Ottawa, Canada.*

In a short note in the *American Journal of Mathematics*, Vol. I, p. 287, I have given Taylor's Theorem under the form

$$f(x+a) = \left(1 - \int_0^a da \cdot \frac{d}{dx}\right)^{-1} fx.$$

I find that although this form is excellent as a merely symbolic notation, the operator $\left(1 - \int_0^a da \cdot \frac{d}{dx}\right)^{-1}$ introduces some obscurity in the reasoning if employed in obtaining the theorem. I therefore propose to show the real method adopted in the note and to apply it to obtain Lagrange's Series.

Taylor's Theorem. Let x and a be independent variables, and fy, $f'y$, ... $f^n y$ be continuous from $y = x$ to $y = x + a$.

Since $\quad \frac{d}{da} f(x+a) = \frac{d}{dx} f(x+a) = f'(x+a)$,

$$\begin{aligned}
\therefore \qquad f(x+a) &= fx + \int_0^a da f'(x+a) \\
&= fx + \int_0^a da \left\{ f'x + \int_0^a da f''(x+a) \right\} \\
&= fx + \frac{a}{1} f'x + \int_0^a da \int_0^a da\, f''(x+a) \\
&= fx + \frac{a}{1} f'x + \frac{a^2}{2} f''x + \left(\int_0^a da\right)^3 f'''(x+a) \\
&\qquad \dots\dots\dots\dots\dots\dots \\
&\qquad \dots\dots\dots\dots\dots\dots \\
&= fx + \frac{a}{1} f'x + \dots\dots + \frac{a^{n-1}}{(n-1)!} f^{n-1}x + \left(\int_0^a da\right)^n f^n(x+a)
\end{aligned}$$

Cayley's Theorem. Let x, a_0, a_1, a_2, ... be independent variables, and $f^m(y + a_{m+1} + a_{m+2} + \ldots\ldots)$ be continuous from $y = x$ to $y = x + a_0 + a_1 + \ldots + a_m$, for all integral values of m from $m = 0$ to $m = n$ inclusive.

Since $\dfrac{d}{da_0} f(x + a_0 + \ldots) = \dfrac{d}{d(a_0 + a_1)} f(x + a_0 + a_1 + \ldots) = \ldots\ldots$

$$= \frac{d}{dx} f(x + a_0 + \ldots) = f'(x + a_0 + a_1 + \ldots),$$

$$\therefore f(x + a_0 + a_1 + \ldots) = f(x + a_1 + a_2 + \ldots) + \int_0^{a_0} da_0 f'(x + a_0 + a_1 + \ldots)$$

$$= f(x + a_1 + a_2 + \ldots)$$

$$+ \int_0^{a_0} da_0 \left\{ f'(x + a_2 + \ldots) + \int_0^{a_0 + a_1} d(a_0 + a_1) f''(x + a_0 + a_1 + \ldots) \right\}$$

$$= f(x + a_1 + a_2 + \ldots) + \frac{[a]^1}{1} f'(x + a_2 + \ldots)$$

$$+ \int_0^{a_0} da_0 \int_0^{a_0 + a_1} d(a_0 + a_1) f''(x + a_0 + a_1 + \ldots)$$

$$= f(x + a_1 + a_2 + \ldots) + \frac{[a]^1}{1} f'(x + a_2 + \ldots) + \frac{[a]^2}{2} f''(x + a_3 + \ldots)$$

$$+ \int_0^{a_0} da_0 \int_0^{a_0 + a_1} d(a_0 + a_1) \int_0^{a_0 + a_1 + a_2} d(a_0 + a_1 + a_2) f'''(x + a_0 + a_1 + \ldots)$$

. . . .

. . . .

$$= f(x + a_1 + a_2 + \ldots) + \frac{[a]^1}{1} f'(x + a_2 + \ldots) + \ldots\ldots$$

$$+ \frac{[a]^{n-1}}{(n-1)!} f^{n-1}(x + a_n + \ldots) + \left[\int_0^a da\right]^n f^n(x + a_0 + a_1 + \ldots)$$

in which

$$\left[\int_0^a da\right]^n U \equiv \int_0^{a_0} da_0 \int_0^{a_0 + a_1} d(a_0 + a_1) \ldots \int_0^{a_0 + a_1 + \ldots + a_{n-1}} d(a_0 + a_1 + \ldots + a_{n-1}) U;$$

and $[a]^n \equiv n! \left[\int_0^a da\right]^n =$ the result of retaining only those terms of the expansion of $(a_0 + a_1 + \ldots\ldots + a_{n-1})^n$ of the form

$Ca_0^\alpha (a_1 + a_2 + \ldots + a_\alpha)^\beta (a_{\alpha+1} + a_{\alpha+2} + \ldots + a_{\alpha+\beta})^\gamma \ldots\ldots$
which $\equiv$ the result of rejecting all terms of the same expansion, of the form

$$Ca_0^{n_0} a_1^{n_1} a_2^{n_2} \ldots\ldots$$

in which $n_{n-1} > 1$, $n_{n-2} + n_{n-1} > 2$, ... $n_1 + n_3 + \ldots + n_{n-1} > n - 1$.

Writing x for $x + a_0 + a_1 + \ldots\ldots$, the theorem becomes

$$fx = f(x - a_0) + \frac{[a]^1}{1} f'(x - a_0 - a_1) + \ldots\ldots$$
$$+ \frac{[a]^{n-1}}{(n-1)!} f^{n-1}(x - a_0 - a_1 - \ldots - a_{n-1}) + R.$$

Let $\{a\}^n \equiv$ the result of retaining only those terms of the expansion of $(a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1})^n$ of the form

$$Ca_0^\alpha a_\alpha^\beta a_{\alpha+\beta}^\gamma \ldots\ldots$$

and affecting each term with the sign $(-1)^{n-m}$ where $m = \frac{\alpha}{\alpha} + \frac{\beta}{\beta} + \frac{\gamma}{\gamma} + \ldots$, and the theorem may be written

$$fx = f(x - a_0) + \frac{\{a\}^1}{1} f'(x - a_1) + \ldots\ldots + \frac{\{a\}^{n-1}}{(n-1)!} f^{n-1}(x - a_{n-1}) + R.$$

At the time of the publication of the above theorem in my note entitled *An Extension of Taylor's Theorem*, (Vol. I, p. 287,) I believed it to be new, but the following note published in *Quart. Journ. Math.*, XIV., 53 (two years before mine), shows the theorem had been given long before by Professor Cayley.

"I wish to put on record the following theorem, given by me as a Senate-House Problem, January, 1851.

If $\{\alpha + \beta + \gamma \ldots\}^p$ denote the expansion of $(\alpha + \beta + \gamma \ldots)^p$, retaining those terms $N\alpha^a\beta^b\gamma^c \ldots$ only in which

$$b + c + d \ldots \not> p - 1, \quad c + d \ldots \not> p - 2, \text{ \&c., \&c.,}$$

then

$$x^n = (x + \alpha)^n - n\{\alpha\}^1(x + \alpha + \beta)^{n-1} + \frac{1}{2}n(n-1)\{\alpha + \beta\}^2(x + \alpha + \beta + \gamma)^{n-2}$$
$$- \frac{1}{6}n(n-1)(n-2)\{\alpha + \beta + \gamma\}^3(x + \alpha + \beta + \gamma + \delta)^{n-3} + \text{\&c.}$$

The theorem, in a somewhat different and imperfectly stated form, is given, Burg, *Crelle*, t. I. (1826), p. 368, as a generalization of Abel's theorem

$$(x + \alpha)^n = x^n + n\alpha(x + \beta)^{n-1} + \frac{1}{2}n(n-1)\alpha(\alpha - 2\beta)(x + 2\beta)^{n-2}$$
$$+ \frac{1}{6}n(n-1)(n-2)\alpha(\alpha - 3\beta)^2(x + 3\beta)^{n-3} + \text{\&c.''}$$

Professor Cayley's proof of this theorem, which we have taken the liberty to call by his name, is given in *Solutions of the Cambridge Senate-House Problems*, 1848–1851, pp. 94–96.

Lagrange's Theorem. Let x and a be independent variables, $u = x + a\phi u$, and fu, $f'u$, ... $f^n u$ continuous from $u = x$ to $u = x + a\phi u$.

Then
$$\frac{dfu}{da} = \phi u \frac{dfu}{dx} = \phi u f'u \frac{du}{dx},$$

$$^{a=0}\left(\frac{d}{da}\right)^n fu = \left(\frac{d}{dx}\right)^{n-1} \{(\phi x)^n f'x\},$$

whence
$$\begin{aligned} fu &= fx + \int_0^a da \left(\frac{dfu}{da}\right) \\ &= fx + \int_0^a da \left\{ \phi u f'u \frac{du}{dx} \right\} \\ &= fx + \int_0^a da \left[\phi x f'x + \int_0^a da \left\{ \frac{d}{da}\left(\phi u f'u \frac{du}{dx}\right) \right\} \right] \\ &= fx + \frac{a}{1} \phi x f'x + \left(\int_0^a da\right)^2 \frac{d}{dx} \left\{ (\phi u)^2 f'u \frac{du}{dx} \right\} \\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ & \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ &= fx + \frac{a}{1} \phi x f'x + \ldots\ldots + \frac{a^{n-1}}{(n-1)!}\left(\frac{d}{dx}\right)^{n-2} \left\{ (\phi x)^{n-1} f'x \right\} \\ & \quad + \left(\int_0^a da\right)^n \left[\left(\frac{d}{dx}\right)^{n-1} \left\{ (\phi u)^n f'u \frac{du}{dx} \right\} \right]. \end{aligned}$$

It may be noticed in this connection that Lagrange's Series can, like Taylor's Theorem, be obtained by integration by parts, and that the remainder can be obtained in the form of a definite integral, thus:—

Let x, a and α be independent variables, and
$$u = x + a\phi u, \qquad v = x + (a - \alpha)\phi v.$$

Then
$$^{\alpha=a}Fv = Fx, \qquad ^{\alpha=0}Fv = Fu,$$

$$\frac{dFv}{da} = -\frac{dFv}{d\alpha}, \text{ and } \frac{d}{da}\left\{ (\phi v)^n \frac{dfv}{dx} \right\} = -\frac{d}{dx}\left\{ (\phi v)^{n+1} \frac{dfv}{dx} \right\}.$$

Therefore
$$fx - fu = \int_0^a d\alpha \left(\frac{d}{d\alpha} fv\right)$$

$$fu = fx + \int_0^a d\alpha \left(\phi v \cdot \frac{dfv}{dx}\right)$$

Integrating by parts,

$$\int_0^a d\alpha\left(\phi v \cdot \frac{dfv}{dx}\right) = a\phi x \,.\, f'x - \int_0^a d\alpha \left\{ \alpha \frac{d}{d\alpha}\left(\phi v \cdot \frac{dfv}{dx}\right)\right\}$$

$$= a\phi x \,.\, f'x + \frac{1}{2}\int_0^a d\alpha \left[\frac{d(\alpha^2)}{d\alpha} \cdot \frac{d}{dx}\left\{(\phi v)^2 \frac{dfv}{dx}\right\}\right].$$

Similarly

$$\frac{1}{2}\int_0^a d\alpha\left[\frac{d(\alpha^2)}{d\alpha}\cdot\frac{d}{dx}\left\{(\phi v)^2\frac{dfv}{dx}\right\}\right] = \frac{a^2}{2}\frac{d}{dx}\left\{(\phi x)^2 f'x\right\}$$

$$+ \frac{1}{2\,.\,3}\int_0^a d\alpha\left[\frac{d(\alpha^3)}{d\alpha}\left(\frac{d}{dx}\right)^2\left\{(\phi v)^3\frac{dfv}{dx}\right\}\right];$$

. . . .

$$\frac{1}{n!}\int_0^a d\alpha\left[\frac{d(\alpha^n)}{d\alpha}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi v)^n\frac{dfv}{dx}\right\}\right] =$$

$$\frac{a^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi x)^n f'x\right\} + \frac{1}{n!}\int_0^a d\alpha\left[\alpha^n\left(\frac{d}{dx}\right)^n\left\{(\phi v)^{n+1}\frac{dfv}{dx}\right\}\right].$$

$$\therefore \quad fu = fx + \frac{a}{1}\phi x \,.\, f'x + \frac{a}{2}\frac{d}{dx}\left\{(\phi x)^2 f'x\right\} + \ldots\ldots$$

$$+ \frac{a^n}{n!}\left(\frac{d}{dx}\right)^{n-1}\left\{(\phi x)^n f'x\right\} + \frac{1}{n!}\int_0^a d\alpha\left[\alpha^n\left(\frac{d}{dx}\right)^n\left\{(\phi v)^{n+1}\frac{dfv}{dx}\right\}\right].$$

This form of the remainder corresponds to that of Taylor's Theorem obtained by integration by parts. (See also xx_1 of the article following this, entitled *Forms of Rolle's Theorem.*) This form can easily be reduced to that obtained by Zolotareff's method, (*Williamson's Integral Calculus*, 3d edition, p. 159), thus:

$$\int_0^a d\alpha\left[\alpha^n\left(\frac{d}{dx}\right)^n\left\{(\phi v)^{n+1}\frac{dfv}{dx}\right\}\right] = \left(\frac{d}{dx}\right)^n\int_0^a d\alpha\left\{(\alpha\,.\,\phi v)^n\,\phi v f'v\,\frac{dv}{dx}\right\}$$

$$= \left(\frac{d}{dx}\right)^n\int_a^0 d\alpha\left\{(a\,.\,\phi v + x - v)^n f'v\,\frac{dv}{d\alpha}\right\}$$

$$= \left(\frac{d}{dx}\right)^n\int_x^u dv\left\{(a\,.\,\phi v + x - v)^n f'v\right\}.$$

P. S.—I find that a form for the remainder in Lagrange's Series was given by Schlömilch, *Liouville*, III_2, 390, (1858.)

Forms of Rolle's Theorem.

BY J. C. GLASHAN, *Ottawa, Canada.*

CONTENTS.

Explanation of Symbols.

$$0 < \theta < 1. \qquad a < \mu < b.$$

$$a;{}^{n}_{m} \equiv a_m + a_{m+1} + \ldots\ldots + a_{n-1}. \qquad a;n \equiv a;{}^{n}_{0}. \qquad \alpha_m \equiv \theta_m a_m.$$

$$\left[\int_0^a da\right]_m^{;\,n} U \equiv \int_0^{a;\,m+1} d(a;m+1) \int_0^{a;\,m+2} d(a;m+2) \ldots \int_0^{a;\,n} d(a;n)\,U.$$

$$\left[\int_0^a da\right]^{;\,n} U \equiv \left[\int_0^a da\right]_0^{;\,n} U.$$

$$[a]_m^n \equiv (n-m)!\left[\int_0^a da\right]_m^{;\,n} = \text{the remainder of the expansion of } (a;n)^{n-m}$$

after rejecting all terms of the form

$$C(a;m+1)^{n_m}\, a_{m+1}^{n_{m+1}}\, a_{m+2}^{n_{m+2}} \ldots\ldots\ldots\ldots,$$

in which

$$\begin{aligned} n_{m+1} + n_{m+2} + \ldots\ldots\ldots\ldots + n_{n-1} &> n-(m+1), \\ n_{m+2} + \ldots\ldots\ldots\ldots + n_{n-1} &> n-(m+2), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ n_{n-1} &> 1. \end{aligned}$$

$$[a]^n \equiv [a]_0^n \equiv n!\left[\int_0^a da\right]^{;\,n}.$$

If u be a complex variable,

$\mathfrak{S} fu \equiv$ the protensive (real) part of fu;

$\mathfrak{V} fu \equiv$ the ditensive (imaginary) part of fu.

I. FUNCTIONS OF A SIMPLE VARIABLE.

1. *Fundamental Form.* (*Rolle's.*) Let Fx and $F'x$ be continuous from $x = a$ to $x = b$, and $Fb - Fa = 0$; then will

$$F'\mu = 0. \qquad \text{i.}$$

A proof of this theorem is given in nearly every text-book on the differential calculus; perhaps the best is that by Professor Mansion in his *Leçons d'Analyse infinitésimale*, (*Gand*).

2. *Extension of Rolle's Form.* If fx, ϕx and $f'x : \phi' x$ be continuous from $x = a$ to $x = b$, and $fb - fa = 0$, then will

$$f'\mu : \phi'\mu = 0. \qquad \text{ii.}$$

This form may be proved by reasoning similar to that employed to prove the Fundamental Form, or it may be deduced from the latter thus:

Let $fx = F\phi x$ and $\mu_1 \equiv \phi a + \theta\,(\phi b - \phi a)$; then, since fx and ϕx are both continuous from $x = a$ to $x = b$, $F\phi x$ will remain continuous while ϕx varies *continuously* from ϕa to ϕb, also $D_{\phi x} F\phi x = D_x F\phi x : D_x \phi x = f'x : \phi' x$ is continuous between the same limits.

Therefore by i. $$D_{\mu_1} F\mu_1 = 0 \qquad (\alpha).$$

But since ϕx is continuous from $x = a$ to $x = b$, although it need not always be intermediate between ϕa and ϕb, yet it must be capable of assuming any proposed intermediate value for at least one value of x between a and b, *i. e.* $\mu_1 = \phi\mu$.

Hence $D_{\mu_1} F\mu_1 = D_{\phi\mu} F\phi\mu = D_\mu F\phi\mu : D_\mu \phi\mu = f'\mu : \phi'\mu, = 0$ by (α).

3. *Lagrange's Form.* Let $F_1 x$ and $F_1' x$ be continuous from $x = a$ to $x = b$, then will

$$F_1 b - F_1 a = (b - a)\, F_1'\mu. \qquad \text{iii.}$$

Let $$Fx \equiv \frac{F_1 x}{F_1 b - F_1 a} - \frac{x}{b - a},$$

then $$F'\mu \equiv \frac{F_1'\mu}{F_1 b - F_1 a} - \frac{1}{b - a} = 0, \text{ by i.};$$

hence $$F_1 b - F_1 a = (b - a)\, F_1'\mu.$$

4. *Cauchy's Form.* Let f_1x and ϕ_1x and either, $(A,)$ $f_1'x$ and ϕ'_1x, $(\phi'_1\mu \neq 0)$ or, $(B,)$ $f_1'x : \phi_1'x$ be continuous from $x = a$ to $x = b$, then will

$$f_1b - f_1a = \frac{f_1'\mu}{\varphi_1'\mu}(\phi_1b - \phi_1a). \qquad \text{iv.}$$

Case A. Let
$$Fx \equiv \frac{f_1x}{f_1b - f_1a} - \frac{\varphi_1x}{\varphi_1b - \varphi_1a},$$

then
$$F'\mu \equiv \frac{f_1'\mu}{f_1b - f_1a} - \frac{\varphi_1'\mu}{\varphi_1b - \varphi_1a} = 0, \text{ by i.};$$

whence, since $\phi_1'\mu \neq 0$,
$$f_1b - f_1a = \frac{f_1'\mu}{\varphi_1'\mu}(\phi_1b - \phi_1a). \qquad \text{iv}_1.$$

Case B. Let
$$fx \equiv \frac{f_1x}{f_1b - f_1a} - \frac{\varphi_1x}{\varphi_1b - \varphi_1a},$$

then
$$\frac{f'\mu}{\varphi_1'\mu} \equiv \frac{f_1'\mu}{\varphi_1'\mu} \cdot \frac{1}{f_1b - f_1a} - \frac{1}{\varphi_1b - \varphi_1a} = 0, \text{ by ii.};$$

whence
$$f_1b - f_1a = \frac{f_1'\mu}{\varphi_1'\mu}(\phi_1b - \phi_1a). \qquad \text{iv}_2.$$

This is an analytical translation of the geometrical proof given by Professor Mansion in his *Leçons*, p. 24, and also in *Messenger of Mathematics*, V_2, 34–35. The analytical demonstrations usually given in the text-books apply only to *Case A.*

5. *Special Case of Cauchy's Form.* Let U_x, V_x, W_x, and either D_xU_x and $D_x(V_x - W_x)$, $[D_\mu(V_\mu - W_\mu) \neq 0]$, or $D_xU_x : D_x(V_x - W_x)$ be all continuous from $x = a$ to $x = b$, then if

$$U_x \equiv f(e+x) + \frac{g-x}{1}f'(e+x) + \ldots\ldots\ldots + \frac{(g-x)^n}{n!}f^n(e+x),$$
$$V_x \equiv \phi(\varepsilon+x) + \frac{l-x}{1}\phi'(\varepsilon+x) + \ldots\ldots\ldots + \frac{(l-x)^p}{p!}\phi^p(\varepsilon+x),$$
$$W_x \equiv \psi(\eta-x) + \frac{x-\lambda}{1}\psi'(\eta-x) + \ldots\ldots\ldots + \frac{(x-\lambda)^q}{q!}\psi^q(\eta-x),$$

it follows immediately from iv. that

$$U_b - U_a = (V_b + W_a - V_a - W_b)\frac{p!\,q!\,(g-\mu)^n f^{n+1}(e+\mu)}{n!\{q!(l-\mu)^p\varphi^{p+1}(\varepsilon+\mu) + p!(\mu-\lambda)^q\psi^{q+1}(\eta-\mu)\}}. \qquad \text{v.}$$

For
$$D_xU_x \equiv \frac{(g-x)^n}{n!}f^{n+1}(e+x), \qquad D_xV_x \equiv \frac{(l-x)^p}{p!}\phi^{p+1}(\varepsilon+x),$$

and
$$D_xW_x \equiv -\frac{(x-\lambda)^q}{q!}\psi^{q+1}(\eta-x).$$

6. *Extension of Cauchy's Form.* Let f_1x, $f_1'x$, f_2x, $f_2'x$, ... f_nx, $f_n'x$, ϕ_1x, $\phi_1'x$, ϕ_2x, $\phi_2'x$, ... ϕ_nx, ϕ'_nx be all continuous from $x=a$ to $x=b$, then will

$$\frac{f'_1\mu}{f_1b-f_1a}+\frac{f'_2\mu}{f_2b-f_2a}+\ldots\ldots+\frac{f'_n\mu}{f_nb-f_na}$$
$$-\left\{\frac{\varphi'_1\mu}{\varphi_1b-\varphi_1a}+\frac{\varphi'_2\mu}{\varphi_2b-\varphi_2a}+\ldots\ldots+\frac{\varphi'_n\mu}{\varphi_nb-\varphi_na}\right\}=0. \qquad \text{vi.}$$

Let $$Fx\equiv\frac{f_1x}{f_1b-f_1a}+\frac{f_2x}{f_2b-f_2a}+\ldots\ldots+\frac{f_nx}{f_nb-f_na}$$
$$-\left\{\frac{\varphi_1x}{\varphi_1b-\varphi_1a}+\frac{\varphi_2x}{\varphi_2b-\varphi_2a}+\ldots\ldots+\frac{\varphi_nx}{\varphi_nb-\varphi_na}\right\},$$

then i. becomes vi.

Cor. If $fx=f_1x=f_2x=\ldots\ldots=f_nx$, then will

$$fb-fa=nf'\mu:\left\{\frac{\varphi'_1\mu}{\varphi_1b-\varphi_1a}+\ldots\ldots+\frac{\varphi'_n\mu}{\varphi_nb-\varphi_na}\right\}. \qquad \text{vii.}$$

Fx being continuous, n must in general be finite.

7. *Forms expressed by Definite Integrals.* Let fx, ϕx, ψx, $\int_c^x fx.\phi xdx$ and $\int_c^x fx.\psi xdx$ be continuous from $x=a$ to $x=b$, and $fx \gtrless 0$ and $\psi x \gtrless 0$, then will

$$\int_a^b fx.\phi xdx=\frac{\varphi\mu}{\psi\mu}\int_a^b fx.\psi xdx. \qquad \text{viii.}$$

Let $$Fx\equiv\frac{\int_c^x fx.\phi xdx}{\int_a^b fx.\phi xdx}-\frac{\int_c^x fx.\psi xdx}{\int_a^b fx.\psi xdx};$$

then $$F'\mu\equiv\frac{f\mu.\phi\mu}{\int_a^b fx.\phi xdx}-\frac{f\mu.\psi\mu}{\int_a^b fx.\psi xdx}=0 \text{ by i.}$$

consequently $$\int_a^b fx.\phi xdx=\frac{\varphi\mu}{\psi\mu}\int_a^b fx.\psi xdx.$$

Cor. If $\psi x=1$, then will

$$\int_a^b fx.\phi xdx=\phi\mu\int_a^b fxdx. \qquad \text{ix.}$$

In viii. and ix. let $\phi x \equiv \int_c^x \phi_1 y dy$ and $\psi x \equiv \int_c^x \psi_1 y dy$

viii. becomes $$\int_a^b fx \int_c^x \phi_1 y dy dx = \frac{\int_c^\mu \phi_1 x dx}{\int_c^\mu \psi_1 x dx} \int_a^b fx \int_c^x \psi_1 y dy dx; \qquad \text{x.}$$

ix. becomes $$\int_a^b fx \int_c^x \phi_1 y dy dx = \int_a^b fx dx . \int_c^\mu \phi_1 x dx. \qquad \text{xi.}$$

Similarly $$\int_a^b fx \int_c^x \phi y \int_c^y \psi z dz dy dx = \int_a^b fx dx . \int_c^\mu \phi x dx . \int_c^{\mu_1} \psi x dx \qquad \text{xii.}$$

where $$\mu_1 = c + \theta(\mu - c).$$

Let $\phi' x \equiv \int \phi x dx$ and let $\phi' c$ be finite. Integrating by parts,

$$\int_a^b fx . \phi x dx = fb . \phi' b - fa . \phi' a - \int_a^b f'x . \phi' x dx$$

$$= fb(\phi' b - \phi' c) - fa(\phi' a - \phi' c) - \int_a^b f'x(\phi' x - \phi' c)\, dx$$

$$= fb \int_c^b \phi x dx - fa \int_c^a \phi x dx - \int_a^b f'x \int_c^x \phi y dy dx \qquad (\beta).$$

by xi., $$= fb \int_c^b \phi x dx - fa \int_c^a \phi x dx - (fb - fa) \int_c^\mu \phi x dx. \qquad \text{xiii.}$$

Let $c = a$, then

$$\int_a^b fx . \phi x dx = fb \int_a^b \phi x dx - \int_a^b f'x \int_a^x \phi y dy dx \qquad (\gamma).$$

$$= fb \int_a^b \phi x dx - (fb - fa) \int_a^\mu \phi x dx. \qquad \text{xiv.}$$

xiv. may also be obtained directly from ix., thus:

$$\int_a^b fx . \phi x dx = fb . \phi' b - fa . \phi' a - \int_a^b f'x . \phi' x dx$$

by ix., $$= fb . \phi' b - fa . \phi' a - (fb - fa)\, \phi' \mu$$

$$= fb \int_\mu^b \phi x dx + fa \int_a^\mu \phi x dx \qquad \text{xiv}_1.$$

and $$= fb \int_a^b \phi x dx - (fb - fa) \int_a^\mu \phi x dx \qquad \text{xiv.}$$

and also $$= fa \int_a^b \phi x dx + (fb - fa) \int_\mu^b \phi x dx. \qquad \text{xiv}_2.$$

The usual proofs of viii. and ix. are substantially the same as Moigno's proof of Cauchy's Theorem, but no hint is at the same time given that iii. and viii. differ only in form of statement. xi., for the case $c = b$, was given by A. Winckler in *Sitzungsberichte der math.-nat. Klasse*, Wien, LX, (1869). (γ) was given by Grunert in *Grunert's Archiv.*, IV, 113. (β) was given by U. H. Meyer in *Grunert's Archiv.*, V, 216. xiv., xiv_1, and xiv_2 were given by Hankel in *Schlömilch's Zeitschrift*, XIV, (1869).

8. *Remainder in Taylor's Theorem.* In the Special Case of Cauchy's Form, let $g = l = b$; $\lambda = a$, $e = o$, $\varepsilon = c - a$, $\eta = k + b$ and $b = a + h$, then v. becomes

$$f(a+h) - \left\{ fa + \frac{h}{1} f'a + \ldots\ldots + \frac{h^n}{n!} f^n a \right\}$$

$$= \Big[\phi(c+h) - \left\{ \phi c + \frac{h}{1} \phi' c + \ldots\ldots + \frac{h^p}{p!} \phi^p c \right\}$$

$$\ldots + \psi(k+h) - \left\{ \psi k + \frac{h}{1} \psi' k + \ldots\ldots + \frac{h^q}{q!} \psi^q k \right\} \Big]$$

$$\times \frac{p!\, q!\, (1-\theta)^n h^n f^{n+1}(a+\theta h)}{n!\, [q!\, (1-\theta)^p h^p \varphi^{p+1}(c+\theta h) + p!\, \theta^q h^q \psi^{q+1}\{k + (1-\theta) h\}]} \equiv R. \qquad \text{xv.}$$

R gives at once all the forms, exclusive of those involving definite integrals, hitherto proposed for the remainder in Taylor's Theorem for a simple variable. Thus, if ψx be constant and $c = a$, R will become the *general* form given in the *Mémoires de l'Académie . . . . . de Montpellier*, V. (1861–1863), and if in addition $p = 0$, R will become the *special* form given in the same memoir. (*Todhunter's Dif. Calc.*, 6[th] *edition*, pp. 404–406). The latter form is also given by Schlömilch in his *Uebungsbuch*, p. 262. If $p = n$, R will become the form given by Professor Mansion in *Messenger of Math.*, V_2, 161.

If ϕx be constant, R will become a form of which the special case for $k = q = 0$ was given by Schlömilch in *Liouville's Journal*, III_2, 384, (1858); also in his *Handbuch*, 1847. The forms of Roche (1858), Cauchy, and Lagrange (1807) are all particular cases of this form of Schlömilch's.

Definite-integral forms of the remainder can be reduced by viii. or ix. above. Thus, to obtain Roche's form in expanding $f(x+h)$ by integration by parts, let the remainder

$$\equiv \frac{1}{n!}\int_0^h v^n f^{n+1}(x+h-v)\,dv$$

and consequently $$= \frac{1}{n!}\int_0^h v^{n-p} f^{n+1}(x+h-v)\,.\,v^p dv$$

by ix. $$= \frac{(\theta h)^{n-p}}{n!} f^{n+1}\{x+(1-\theta)h\}\int_0^h v^p dv$$

$$= \frac{\theta^{n-p}h^{n+1}}{n!\,(p+1)} f^{n+1}\{x+(1-\theta)h\}.$$

More complicated forms of the remainder can be obtained by employing the Extension of Cauchy's Form given in 6. One of these has been noticed by Professor Mansion, *Mess. of Math.*, V_2, 162.

9. *Remainder in Cayley's Theorem.* In the note preceding this article, entitled *Simple and Uniform Method of Obtaining Taylor's, Cayley's, and Lagrange's Series*, it is shown that

$$f(x+a;m) = f\left(x+a;{m \atop 1}\right) + \frac{[a]^1}{1!} f'\left(x+a;{m \atop 2}\right) + \ldots$$

$$\ldots + \frac{[a]^{n-1}}{(n-1)!} f^{n-1}\left(x+a;{m \atop n}\right) + R \qquad \text{xvi.}$$

in which $R \equiv \left[\int_0^a da\right]^{;n} f^n(x+a;m)$ (δ).

by ix. $$= \left[\int_0^a da\right]^{;n-1}\left(f^n\left\{x+\theta'(a;n)+a;{m \atop n}\right\}\int_0^{a;n} d(a;n)\right)$$

$$= \left[\int_0^a da\right]^{;n-1}\left(f^n\left\{x+\theta'(a;n)+a;{m \atop n}\right\}[a]^n_{n-1}\right)$$

by ix., $$= \left[\int_0^a da\right]^{;n-2}\left(\frac{[a]^n_{n-2}}{2!} f^n\left\{x+\theta''\theta_n(a;n-1)+\alpha_{n-1}+a;{m \atop n}\right\}\right)$$

. . . .

. . . .

$$= \frac{[a]^n}{n!} f^n\left(x+\alpha;n+a;{m \atop n}\right). \qquad (\epsilon).$$

Substitute x for $x+a;m$ and θ for $1-\theta$ and the above becomes

$$fx = f(x-a;1) + \frac{[a]^1}{1} f'(x-a;2) + \ldots\ldots + \frac{[a]^{n-1}}{(n-1)!} f^{n-1}(x-a;n)$$

$$+ \frac{[a]^n}{n!} f^n(x-\alpha;n) \qquad \text{xvii.}$$

In III. of *An Extension of Taylor's Theorem*, (this Journal, Vol. I, p. 287), the remainder

$$\equiv\Big[\int_0^a da\Big]^{;n}\Big\{1-\int_0^{a;n+1} d(a;n+1)\,.\,\frac{d}{dx}\Big\}^{-1} f^n\Big(x+a;\,{}^{m}_{n+1}\Big)$$

and by I. of the same note, this

$$\equiv\Big[\int_0^a da\Big]^{;n} f^n(x+a;m)$$

which is (δ) given above.

10. *Remainder in Lagrange's Series.* Let x and y be independent variables, $w=x+y\phi w$, $v=x+a\phi v$, $u=x+b\phi u$ and $u_1=x+\mu\phi u_1$; also let fw, $\frac{dfw}{dy}, \ldots\ldots \frac{d^{n+1}w}{dy^{n+1}}$, $\psi(c-y)$, $\psi'(c-y)$ be all continuous from $y=a$ to $y=b$, and $\psi'(c-\mu)\gtrless 0$.

Since
$$\frac{d}{dy}\Big\{(\phi w)^n\frac{dfw}{dx}\Big\}=\frac{d}{dx}\Big\{(\phi w)^{n+1}\frac{dfw}{dx}\Big\},$$

if $Fy\equiv fw+\frac{k-y}{1}\phi w\frac{dfw}{dx}+\ldots\ldots+\frac{(k-y)^n}{n!}\Big(\frac{d}{dx}\Big)^{n-1}\Big\{(\phi w)^n\frac{dfw}{dx}\Big\}$,

then $F'y\equiv\frac{(k-y)^n}{n!}\Big(\frac{d}{dx}\Big)^n\Big\{(\phi w)^{n+1}\frac{dfw}{dx}\Big\}$,

and, by iv.,
$$Fb=Fa+\frac{\psi(c-a)-\psi(c-b)}{\psi'(c-\mu)}F'\mu;$$

or, writing this in full,

$$fu+\frac{k-b}{1}\phi u\frac{dfu}{dx}+\ldots\ldots\ldots+\frac{(k-b)^n}{n!}\Big(\frac{d}{dx}\Big)^{n-1}\Big\{(\phi u)^n\frac{dfu}{dx}\Big\}=$$
$$fv+\frac{k-a}{1}\phi v\frac{dfv}{dx}+\ldots\ldots\ldots+\frac{(k-a)^n}{n!}\Big(\frac{d}{dx}\Big)^{n-1}\Big\{(\phi v)^n\frac{dfv}{dx}\Big\}$$
$$+\frac{\psi(c-a)-\psi(c-b)}{\psi'(c-\mu)}\cdot\frac{(k-\mu)^n}{n!}\Big(\frac{d}{dx}\Big)^n\Big\{(\phi u_1)^{n+1}\frac{dfu_1}{dx}\Big\}. \qquad \text{xviii.}$$

If $k=b$, $c=b+h$ and $a=0$ and therefore $v=x$ and $u_1=x+\theta b\phi u_1$, this reduces to Lagrange's Series, or

$$fu=fx+\frac{b}{1}\phi x.f'x+\ldots\ldots\ldots+\frac{b^n}{n!}\Big(\frac{d}{dx}\Big)^{n-1}\{(\phi x)^n f'x\}$$
$$+\frac{\psi(h+b)-\psi h}{\psi'\{h+(1-\theta)b\}}\cdot\frac{\{(1-\theta)b\}^n}{n!}\Big(\frac{d}{dx}\Big)^n\Big\{(\phi u_1)^{n+1}\frac{dfu_1}{dx}\Big\}. \qquad \text{xix.}$$

Let $h = 0$ and $\psi b = b^{p+1}$, then the remainder will become

$$\frac{(1-\theta)^{n-p} b^{n+1}}{n!\,(p+1)} \left(\frac{d}{dx}\right)^n \left\{ (\phi u_1)^{n+1} \frac{dfu_1}{dx} \right\}. \qquad \text{xx.}$$

For $\psi(c-y)$ write $\psi_1(y+h)$, which may be done since ψ and ψ_1 are both arbitrary, then making $k = b$ and $a = 0$, xviii. becomes

$$fu = fx + \ .\ .\ . + \frac{\psi_1(c+b) - \psi_1 c}{\psi'_1(c+\theta b)} \cdot \frac{\{(1-\theta)b\}^n}{n!} \left(\frac{d}{dx}\right)^n \left\{ (\phi u_1)^{n+1} \frac{dfu_1}{dx} \right\}. \qquad \text{xxi.}$$

Again for $\psi(c-y)$ and $\psi_1(y+h)$, there may be substituted functions of the form of W_x, V_x, or $V_x - W_x$ of 5, or functions bearing the same relation to Lagrange's Series that these do to Taylor's.

In the note *Simple and Uniform Method*, &c., it is shown that if $u = x + b\phi u$ and $v = x + (b-\beta)\phi v$ the remainder in Lagrange's Series is

$$\frac{1}{n!}\int_0^b d\beta \left[\beta^n \left(\frac{d}{dx}\right)^n \left\{ (\phi v)^{n+1} \frac{dfv}{dx} \right\} \right]$$

which

$$= \frac{1}{n!}\int_0^b d\beta \left[\beta^{n-p} \left(\frac{d}{dx}\right)^n \left\{ (\phi v)^{n+1} \frac{dfv}{dx} \right\} \beta^p \right]$$

by ix.

$$= \frac{(1-\theta)^{n-p} b^{n+1}}{n!\,(p+1)} \left(\frac{d}{dx}\right)^n \left\{ (\phi u_1)^{n+1} \frac{dfu_1}{dx} \right\} \qquad \text{xx}_1.$$

in which $u_1 = x + \theta b \phi u_1$. This is the same form of remainder as that found in xx. Had viii. been used instead of ix. in the reduction, forms like those of xix. or xxi. would have been obtained.

II. Functions of a Complex Variable.

11. *General Form.* Let $u = x + i\phi x$, $u_0 = x_0 + i\phi x_0$, $u_1 = x_1 + i\phi x_1$, $v_1 = u_0 + \theta_1(u_1 - u_0)$, and $v_2 = u_0 + \theta_2(u_1 - u_0)$. If ϕx and $\phi' x$ remain continuous while x varies continuously from $x = x_0$ to $x = x_1$, if also Φu and $\Phi' u$ remain continuous from $u = u_0$ to $u = u_1$, and if $\Phi u_1 - \Phi u_0 = 0$, then will

$$\mathfrak{R}\Phi' v_1 = 0 \quad \text{and} \quad \mathfrak{I}\Phi' v_2 = 0. \qquad \text{xxii.}$$

12. *Special Form.* Let $w = (u - u_0) : (u_1 - u_0)$ and

$$\Phi u \equiv (u_1 - u_0)\{(Fu_1 - Fu_0)(\Psi w - \Psi 0) - (Fu - Fu_0)(\Psi 1 - \Psi 0)\}$$

and $\therefore\ \Phi' u \equiv (Fu_1 - Fu_0)\Psi' w - (u_1 - u_0)(\Psi 1 - \Psi 0) F' u$

$$\therefore\ Fu_1 - Fu_0 = \frac{\Psi_1 1 - \Psi_1 0}{\Psi'_1 \theta_1} \mathfrak{R}\{(u_1 - u_0) F' v_1\} + \frac{\Psi_2 1 - \Psi_2 0}{\Psi'_2 \theta_2} \mathfrak{I}\{(u_1 - u_0) F' v_2\}. \text{ xxiii.}$$

13. *Remainder in Taylor's Thoerem.* In the preceding form let

$$Fu \equiv f(h+lu) + \frac{k-lu}{1} f'(h+lu) + \ldots\ldots + \frac{(k-lu)^n}{n!} f^n(h+lu),$$

$$\Psi w \equiv \psi(a+bw) + \frac{c-bw}{1} \psi'(a+bw) + \ldots\ldots + \frac{(c-bw)^m}{m!} \psi^m(a+bw),$$

$$\therefore f(h+lu_1) + \frac{k-lu_1}{1} f'(h+lu_1) + \ldots\ldots + \frac{(k-lu_1)^n}{n!} f^n(h+lu_1)$$

$$- \left\{ f(h+lu_0) + \frac{k-lu_0}{1} f'(h+lu_0) + \ldots\ldots + \frac{(k-lu_0)^n}{n!} f^n(h+lu_0) \right\}$$

$$= \Big[\psi_1(a+b) + \frac{c-b}{1} \psi_1'(a+b) + \ldots\ldots + \frac{(c-b)^p}{p!} \psi_1^p(a+b)$$

$$- \left\{ \psi_1 a + \frac{c}{1} \psi_1' a + \ldots\ldots + \frac{c^p}{p!} \psi_1^p a \right\} \Big] R_1$$

$$+ \Big[\psi_2(a+\beta) + \frac{\gamma-\beta}{1} \psi_2'(a+\beta) + \ldots\ldots + \frac{(\gamma-\beta)^q}{q!} \psi_2^q(a+\beta)$$

$$- \left\{ \psi_2 a + \frac{\gamma}{1} \psi_2' a + \ldots\ldots + \frac{\gamma^q}{q!} \psi_2^q a \right\} \Big] R_2 \equiv R. \qquad \text{xxiv.}$$

in which $$R_1 \equiv \frac{p!\, l\, \mathfrak{S}\{(u_1-u_0)(k-lv_1)^n f^{n+1}(h+lv_1)\}}{n!\, b\, (c-\theta_1 b)^p \psi_1^{p+1}(a+\theta_1 b)}, \qquad (\zeta_1).$$

and $$R_2 \equiv \frac{q!\, l\, \mathfrak{V}\{(u_1-u_0)(k-lv_2)^n f^{n+1}(h+lv_2)\}}{n!\, \beta\, (\gamma-\theta_2\beta)^q \psi_2^{q+1}(a+\theta_2\beta)}. \qquad (\zeta_2).$$

Had it been assumed that

$$\Psi w \equiv \Psi_1(a_1+b_1w) - \Psi_2(a_2-b_2w) + \frac{c_1-b_1w}{1} \Psi_1'(a_1+b_1w) - \frac{c_2-b_2w}{1} \Psi_2'(a_2-b_2w) + \&\text{c.},$$

a general theorem would have been obtained of which v. is the form for a real variable in the particular case $l = b_1 = 1$.

Taylor's theorem with Remainder is the particular case of xxiv. for $h=0$, $k=u_1$, and $l=1$. If in addition to these limitations, $c=b$, $\gamma=\beta$, and $u_1-u_0=t$, xxiv. becomes (writing u for u_0)

$$f(u+t) - \left\{ fu + \frac{t}{1} f'u + \ldots\ldots + \frac{t^n}{n!} f^n u \right\}$$

$$= \Big[\psi_1(a+b) - \left\{ \psi_1 a + \frac{b}{1} \psi_1' a + \ldots\ldots + \frac{b^p}{p!} \psi_1^p a \right\} \Big] R_1$$

$$+ \Big[\psi_2(a+\beta) - \left\{ \psi_2 a + \frac{\beta}{1} \psi_2' + a \ldots\ldots + \frac{\beta^q}{q!} \psi_2^q a \right\} \Big] R_2 \qquad \text{xxv.}$$

in which $$R_1 \equiv \frac{p!\,(1-\theta_1)^{n-p}\mathfrak{S}\{t^{n+1}f^{n+1}(u+\theta_1 t)\}}{n!\,b^{p+1}\psi_1^{p+1}(a+\theta_1 b)}, \qquad (\eta_1).$$

and $$R_2 \equiv \frac{q!\,(1-\theta_2)^{n-q}\mathfrak{V}\{t^{n+1}f^{n+1}(u+\theta_2 t)\}}{n!\,\beta^{q+1}\psi_2^{q+2}(\alpha+\theta_1\beta)}. \qquad (\eta_2).$$

See *Elementary Demonstration of Taylor's Theorem for Functions of an Imaginary Variable*, by Professor Mansion, *Mess. of Math.* VIII_2. pp. 17–20 (1878).

In this note I have spoken of *Forms of Rolle's Theorem* because in reality the various theorems differ merely in form, not at all in generality. Some forms are more convenient than others, and exhibit explicitly what the others contain implicitly. Thus Cauchy's Form is not more general than that of Lagrange, but only sometimes more convenient; every theorem that can be proved by the former can likewise be proved by the latter.

On the 8-Square Imaginaries.

BY PROFESSOR CAYLEY.

I write throughout 0 to denote positive unity; and uniting with it the seven imaginaries $1, \ldots 7$, form an octavic system $0, 1, 2, 3, 4, 5, 6, 7$, the laws of combination being

$$
\begin{array}{lll}
\multicolumn{3}{c}{0^2=0,\ 1^2=2^2=3^2=4^2=5^2=6^2=7^2=-0,} \\
123=\varepsilon_1, & 145=\varepsilon_2, & 167=\varepsilon_3, \\
 & 246=\varepsilon_4, & 257=\varepsilon_5, \\
 & 347=\varepsilon_6, & 356=\varepsilon_7,
\end{array}
$$

where $\varepsilon = \pm$, viz. each ε has a determinate value $+$ or $-$ as the case may be; and where the formula, $123=\varepsilon_1$, denotes the six equations

$$
\begin{array}{lll}
23=\ \ \varepsilon_1 1, & 31=\ \ \varepsilon_1 2, & 12=\ \ \varepsilon_1 3, \\
32=-\varepsilon_1 1, & 13=-\varepsilon_1 2, & 21=-\varepsilon_1 3,
\end{array}
$$

and so for the other formulæ: the multiplication table of the eight symbols thus is

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	-0	$\varepsilon_1 3$	$-\varepsilon_1 2$	$\varepsilon_2 5$	$-\varepsilon_2 4$	$\varepsilon_3 7$	$-\varepsilon_3 6$
2	2	$-\varepsilon_1 3$	-0	$\varepsilon_1 1$	$\varepsilon_4 6$	$\varepsilon_5 7$	$-\varepsilon_4 4$	$-\varepsilon_5 5$
3	3	$\varepsilon_1 2$	$-\varepsilon_1 1$	-0	$\varepsilon_6 7$	$\varepsilon_7 6$	$-\varepsilon_7 5$	$-\varepsilon_6 4$
4	4	$-\varepsilon_2 5$	$-\varepsilon_4 6$	$-\varepsilon_6 7$	-0	$\varepsilon_2 1$	$\varepsilon_4 2$	$\varepsilon_6 3$
5	5	$\varepsilon_2 4$	$-\varepsilon_5 7$	$-\varepsilon_7 6$	$-\varepsilon_2 1$	-0	$\varepsilon_7 3$	$\varepsilon_5 2$
6	6	$-\varepsilon_3 7$	$\varepsilon_4 4$	$\varepsilon_7 5$	$-\varepsilon_4 2$	$-\varepsilon_7 3$	-0	$\varepsilon_3 1$
7	7	$\varepsilon_3 6$	$\varepsilon_5 5$	$\varepsilon_6 4$	$-\varepsilon_6 3$	$-\varepsilon_5 2$	$-\varepsilon_3 1$	-0

Hence if 0, 1, 2, 3, 4, 5, 6, 7 and 0′, 1′, 2′, 3′, 4′, 5′, 6′, 7′ denote ordinary algebraical magnitudes, and we form the product

$$(0\mathbf{0}+1\mathbf{1}+2\mathbf{2}+3\mathbf{3}+4\mathbf{4}+5\mathbf{5}+6\mathbf{6}+7\mathbf{7})(0'\mathbf{0}+1'\mathbf{1}+2'\mathbf{2}+3'\mathbf{3}+4'\mathbf{4}+5'\mathbf{5}+6'\mathbf{6}+7'\mathbf{7}),$$

this is at once found to be =

$$\begin{array}{ll} (00'-11'-22'-33'-44'-55'-66'-77') & \mathbf{0} \\ +\,(01'+0'1+\varepsilon_1 23+\varepsilon_2 45+\varepsilon_3 67) & \mathbf{1} \\ +\,(02'+0'2+\varepsilon_1 31+\varepsilon_4 46+\varepsilon_5 57) & \mathbf{2} \\ +\,(03'+0'3+\varepsilon_1 12+\varepsilon_6 47+\varepsilon_7 56) & \mathbf{3} \\ +\,(04'+0'4+\varepsilon_1 51+\varepsilon_4 62+\varepsilon_6 73) & \mathbf{4} \\ +\,(05'+0'5+\varepsilon_2 14+\varepsilon_5 72+\varepsilon_7 63) & \mathbf{5} \\ +\,(06'+0'6+\varepsilon_3 71+\varepsilon_4 24+\varepsilon_7 35) & \mathbf{6} \\ +\,(07'+0'7+\varepsilon_3 16+\varepsilon_5 25+\varepsilon_6 34) & \mathbf{7} \end{array}$$

where 12 is written to denote $12'-1'2$, and so in other cases.

The sum of the squares of the eight coefficients of $\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}$ respectively, will, if certain terms destroy each other, be

$$=(0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2)\,(0'^2+1'^2+2'^2+3'^2+4'^2+5'^2+6'^2+7'^2)$$

viz., the sum of the squares contains the several terms

$$\begin{array}{lllllll} \varepsilon_1\varepsilon_2 23.45, & \varepsilon_1\varepsilon_3 23.67, & \varepsilon_1\varepsilon_4 31.46, & \varepsilon_1\varepsilon_5 31.57, & \varepsilon_1\varepsilon_6 12.47, & \varepsilon_1\varepsilon_7 12.56, & \varepsilon_2\varepsilon_3 45.67, \\ \varepsilon_4\varepsilon_7 24.35, & \varepsilon_4\varepsilon_6 62.73, & \varepsilon_2\varepsilon_7 14.63, & \varepsilon_2\varepsilon_6 51.73, & \varepsilon_2\varepsilon_5 14.72, & \varepsilon_2\varepsilon_4 51.62, & \varepsilon_4\varepsilon_5 46.57, \\ \varepsilon_5\varepsilon_6 25.34, & \varepsilon_5\varepsilon_7 72.63, & \varepsilon_3\varepsilon_6 16.34, & \varepsilon_3\varepsilon_7 71.35, & \varepsilon_3\varepsilon_4 71.24, & \varepsilon_3\varepsilon_5 16.25, & \varepsilon_6\varepsilon_7 47.56, \end{array}$$

and observing that $21=-12$ etc., and that we have identically $23\cdot45+24\cdot53+25\cdot34=$ zero, etc., then the three terms of each column will vanish, provided a proper relation exists between the ε's, viz., the conditions which we thus obtain are

$$\begin{array}{lllll} \varepsilon_1\varepsilon_2 = & -\varepsilon_4\varepsilon_7 = & & \varepsilon_5\varepsilon_6, \\ \varepsilon_1\varepsilon_3 = & -\varepsilon_4\varepsilon_6 = & & \varepsilon_5\varepsilon_7, \\ \varepsilon_1\varepsilon_4 = & -\varepsilon_3\varepsilon_6 = & - & \varepsilon_2\varepsilon_7, \\ \varepsilon_1\varepsilon_5 = & \varepsilon_3\varepsilon_7 = & & \varepsilon_2\varepsilon_6, \\ \varepsilon_1\varepsilon_6 = & \varepsilon_2\varepsilon_5 = & - & \varepsilon_3\varepsilon_4, \\ \varepsilon_1\varepsilon_7 = & -\varepsilon_2\varepsilon_4 = & & \varepsilon_3\varepsilon_5, \\ \varepsilon_2\varepsilon_3 = & -\varepsilon_4\varepsilon_5 = & & \varepsilon_6\varepsilon_7. \end{array}$$

We may without loss of generality assume $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$; the equations then become

$$\begin{array}{lllll}
+ & = & -\,\varepsilon_4\varepsilon_7 & = & \varepsilon_5\varepsilon_6 \\
+ & = & -\,\varepsilon_4\varepsilon_6 & = & \varepsilon_5\varepsilon_7 \\
+ & = & -\,\varepsilon_4\varepsilon_5 & = & \varepsilon_6\varepsilon_7 \\
\varepsilon_4 & = & -\,\varepsilon_6 & = & -\,\varepsilon_7 \\
\varepsilon_5 & = & \varepsilon_7 & = & \varepsilon_6 \\
\varepsilon_6 & = & \varepsilon_5 & = & -\,\varepsilon_4 \\
\varepsilon_7 & = & -\,\varepsilon_4 & = & \varepsilon_5
\end{array}$$

and writing $\theta = \pm$ at pleasure, these are all satisfied if $-\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \theta$. The terms written down all disappear, and the sum of the squares of the eight coefficients thus becomes equal to the product of two sums each of them of eight squares, viz., this is the case if $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$, $-\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \theta$, θ being $= \pm$ at pleasure: the resulting system of imaginaries may be said to be an 8-square system.

We may inquire whether the system is associative; for this purpose, supposing in the first instance that the ε's remain arbitrary, we form the complete system of the values of the triplets $12\cdot3$, $1\cdot23$, etc. (read the top line $12\cdot3 = -\varepsilon_1 0$; $1\cdot23 = -\varepsilon_1 0$, the next line $12\cdot4 = \varepsilon_1\varepsilon_6 7$, $1\cdot24 = \varepsilon_3\varepsilon_4 7$, and so in other cases):

$12\cdot3=$	$1\cdot23=$	$-\varepsilon_1$	$-\varepsilon_1$	0
$12\cdot4=$	$1\cdot24=$	$\varepsilon_1\varepsilon_6$	$\varepsilon_3\varepsilon_4$	7
$12\cdot5=$	$1\cdot25=$	$\varepsilon_1\varepsilon_7$	$-\varepsilon_3\varepsilon_5$	6
$12\cdot6=$	$1\cdot26=$	$-\varepsilon_1\varepsilon_7$	$-\varepsilon_2\varepsilon_4$	5
$12\cdot7=$	$1\cdot27=$	$-\varepsilon_1\varepsilon_6$	$\varepsilon_2\varepsilon_5$	4
$13\cdot4=$	$1\cdot34=$	$-\varepsilon_1\varepsilon_4$	$-\varepsilon_3\varepsilon_6$	6
$13\cdot5=$	$1\cdot35=$	$-\varepsilon_1\varepsilon_5$	$\varepsilon_3\varepsilon_7$	7
$13\cdot6=$	$1\cdot36=$	$\varepsilon_1\varepsilon_4$	$\varepsilon_2\varepsilon_7$	4
$13\cdot7=$	$1\cdot37=$	$\varepsilon_1\varepsilon_5$	$-\varepsilon_2\varepsilon_6$	5
$14\cdot5=$	$1\cdot45=$	$-\varepsilon_2$	$-\varepsilon_2$	0
$14\cdot6=$	$1\cdot46=$	$\varepsilon_2\varepsilon_7$	$\varepsilon_1\varepsilon_4$	3
$14\cdot7=$	$1\cdot47=$	$\varepsilon_2\varepsilon_5$	$-\varepsilon_1\varepsilon_6$	2
$15\cdot6=$	$1\cdot56=$	$-\varepsilon_2\varepsilon_4$	$-\varepsilon_1\varepsilon_7$	2
$15\cdot7=$	$1\cdot57=$	$-\varepsilon_2\varepsilon_6$	$\varepsilon_1\varepsilon_5$	3
$16\cdot7=$	$1\cdot67=$	$-\varepsilon_3$	$-\varepsilon_3$	0
$23\cdot4=$	$2\cdot34=$	$\varepsilon_1\varepsilon_2$	$-\varepsilon_5\varepsilon_6$	5
$23\cdot5=$	$2\cdot35=$	$-\varepsilon_1\varepsilon_2$	$-\varepsilon_4\varepsilon_7$	4
$23\cdot6=$	$2\cdot36=$	$\varepsilon_1\varepsilon_3$	$-\varepsilon_5\varepsilon_7$	7
$23\cdot7=$	$2\cdot37=$	$-\varepsilon_1\varepsilon_3$	$-\varepsilon_4\varepsilon_6$	6
$24\cdot5=$	$2\cdot45=$	$-\varepsilon_4\varepsilon_7$	$-\varepsilon_1\varepsilon_2$	3
$24\cdot6=$	$2\cdot46=$	$-\varepsilon_4$	$-\varepsilon_4$	0
$24\cdot7=$	$2\cdot47=$	$\varepsilon_3\varepsilon_4$	$\varepsilon_1\varepsilon_6$	1
$25\cdot6=$	$2\cdot56=$	$-\varepsilon_3\varepsilon_5$	$\varepsilon_1\varepsilon_7$	1
$25\cdot7=$	$2\cdot57=$	$-\varepsilon_5$	$-\varepsilon_5$	0
$26\cdot7=$	$2\cdot67=$	$-\varepsilon_4\varepsilon_6$	$-\varepsilon_1\varepsilon_3$	3
$34\cdot5=$	$3\cdot45=$	$-\varepsilon_5\varepsilon_6$	$\varepsilon_1\varepsilon_2$	2
$34\cdot6=$	$3\cdot46=$	$-\varepsilon_3\varepsilon_6$	$-\varepsilon_1\varepsilon_4$	1
$34\cdot7=$	$3\cdot47=$	$-\varepsilon_6$	$-\varepsilon_6$	0
$35\cdot6=$	$3\cdot56=$	$-\varepsilon_7$	$-\varepsilon_7$	0
$35\cdot7=$	$3\cdot57=$	$\varepsilon_3\varepsilon_7$	$-\varepsilon_1\varepsilon_5$	1
$36\cdot7=$	$3\cdot67=$	$-\varepsilon_5\varepsilon_7$	$\varepsilon_1\varepsilon_3$	2
$45\cdot6=$	$4\cdot56=$	$\varepsilon_2\varepsilon_3$	$-\varepsilon_6\varepsilon_7$	7
$45\cdot7=$	$4\cdot57=$	$-\varepsilon_2\varepsilon_3$	$-\varepsilon_4\varepsilon_5$	6
$46\cdot7=$	$4\cdot67=$	$-\varepsilon_4\varepsilon_5$	$-\varepsilon_2\varepsilon_3$	5
$56\cdot7=$	$5\cdot67=$	$-\varepsilon_6\varepsilon_7$	$\varepsilon_2\varepsilon_3$	4

Write as before $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$; then disregarding the lines (such as the first line) which contain the symbol 0, and writing down only the signs as given in the third and fourth columns, these are

ε_6	ε_4	ε_5	$-\varepsilon_6$	$+$	$-\varepsilon_5\varepsilon_7$	$-\varepsilon_6$	$-\varepsilon_4$
ε_7	$-\varepsilon_5$	ε_7	ε_4	$-$	$-\varepsilon_4\varepsilon_6$	ε_7	$-\varepsilon_5$
$-\varepsilon_7$	$-\varepsilon_4$	ε_5	$-\varepsilon_6$	$-\varepsilon_4\varepsilon_7$	$-$	$-\varepsilon_5\varepsilon_7$	$+$
$-\varepsilon_6$	ε_5	$-\varepsilon_4$	$-\varepsilon_7$	ε_4	ε_6	$+$	$-\varepsilon_6\varepsilon_7$
$-\varepsilon_4$	$-\varepsilon_6$	$-\varepsilon_6$	ε_5	$-\varepsilon_5$	ε_7	$-$	$-\varepsilon_4\varepsilon_5$
$-\varepsilon_5$	ε_7	$+$	$-\varepsilon_5\varepsilon_6$	$-\varepsilon_4\varepsilon_6$	$-$	$-\varepsilon_4\varepsilon_5$	$-$
ε_4	ε_7	$-$	$-\varepsilon_4\varepsilon_7$	$-\varepsilon_5\varepsilon_6$	$+$	$-\varepsilon_6\varepsilon_7$	$+$

and we hence see at once that the pairs of signs in the two columns respectively cannot be made identical: to make them so, we should have $\varepsilon_6 = \varepsilon_4$, $\varepsilon_7 = -\varepsilon_5$, $\varepsilon_7 = \varepsilon_4$, that is $\varepsilon_4 = \varepsilon_6 = \varepsilon_7 = -\varepsilon_5$, which is inconsistent with the last equation of the system $-\varepsilon_6\varepsilon_7 = +$. Hence the imaginaries $1, 2, 3, 4, 5, 6, 7$, as defined by the original conditions, are not in any case associative.

If we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$ and also $-\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \theta$, that is, if the imaginaries belong to the 8-square formula, then it is at once seen that each pair consists of two opposite signs; that is, for the several triads 123, 145, 167, 246, 257, 347, 356 used for the definition of the imaginaries, the associative property holds good, $12 \cdot 3 = 1 \cdot 23$, etc.; but for each of the remaining twenty-eight triads, *the two terms are equal but of opposite signs*, viz. $12 \cdot 4 = -1 \cdot 24$, etc.; so that the product 124 of any such three symbols has no determinate meaning.

BALTIMORE, *March 5th*, 1882.

On Certain Metrical Properties of Surfaces.

By Thomas Craig, *Johns Hopkins University.*

First consider a surface in a space of $n+1$ dimensions. For brevity in speaking of spaces of any number of dimensions I shall use the symbol (commonly employed) M_{n+1}; this is taken to denote a space of $n+1$ dimensions. A surface in M_{n+1} is understood to be expressed by a single relation between the $n+1$ coördinates which determine the position of a point in M_{n+1}. Denoting these coördinates by x_i where $i=1, 2, 3 \ldots n+1$, the equation

$$\Phi(x_1 x_2 \ldots x_{n+1}) = 0$$

is the equation of an n-dimensional surface in M_{n+1}. Since we have one relation connecting the $n+1$ quantities x_i, each of these may be given as a function of n independent variables $u_1 u_2 \ldots u_n$.

Denote by $a_k^{(i)}$ the differential coefficient of x_i with respect to u_k, then

$$dx_i = a_1^{(i)}du_1 + a_2^{(i)}du_2 + \ldots + a_n^{(i)}du_n;$$

also denote by $a_{jk}^{(i)}$ the second differential coefficient of x_i with respect to u_j and u_k, then

$$d^2x_i = a_{11}^{(i)}du_1^2 + 2a_{12}^{(i)}du_1du_2 + 2a_{13}^{(i)}du_1du_3 + \ldots + a_{nn}^{(i)}du_n^2.$$

For the element of length ds of a curve traced on $\Phi = 0$, we have then

$$ds^2 = \sum_{i=1}^{i=n+1} a_1^{(i)^2}du_1^2 + \Sigma a_2^{(i)^2}du_2^2 + \ldots + \Sigma a_n^{(i)^2}du_n^2$$
$$+ 2\Sigma a_1^{(i)}a_2^{(i)}du_1du_2 + \ldots + 2\Sigma a_{n-1}^{(i)}a_n^{(i)}du_{n-1}du_n.$$

The limits of the summation have only been written once, as they are of course the same for every term. For brevity, write

$$E_{ii} = a_i^{(1)^2} + a_i^{(2)^2} + \ldots + a_i^{(n+1)^2}$$
$$E_{jk} = a_j^{(1)}a_k^{(1)} + a_j^{(2)}a_k^{(2)} + \ldots + a_j^{(n)}a_k^{(n)};$$

then for ds we have

$$ds^2 = E_{11}du_1^2 + E_{12}du_1du_2 + \ldots + E_{jk}du_jdu_k + \ldots + E_{nn}du_n^2.$$

From the quantities $a_j^{(i)}$ form the determinants of which

$$\begin{vmatrix} a_1^{(2)} & a_1^{(3)} & \ldots & a_1^{(n+1)} \\ a_2^{(2)} & a_2^{(3)} & \ldots & a_2^{(n+1)} \\ \ldots & \ldots & \ldots & \ldots \\ a_n^{(2)} & a_n^{(3)} & \ldots & a_n^{(n+1)} \end{vmatrix} = A_1$$

is the first. There will be $n+1$ of these determinants each of the degree n; squaring these and adding, we obtain, as is well known, a symmetrical determinant of the degree n, viz.

$$V^2 = \sum_1^n A_i^2 = \begin{vmatrix} E_{11} & E_{12} & E_{13} & \ldots & E_{1n} \\ E_{21} & E_{22} & E_{23} & \ldots & E_{2n} \\ E_{31} & E_{32} & E_{33} & \ldots & E_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ E_{n1} & E_{n2} & E_{n3} & \ldots & E_{nn} \end{vmatrix};$$

of course $E_{ij} = E_{ji}$.

The differential equation of the surface $\Phi = 0$ can now be written as

$$A_1dx_1 + A_2dx_2 + \ldots + A_{n+1}dx_{n+1} = 0.$$

The direction-cosines of the normal to this surface at any point are

$$\alpha_1,\ \alpha_2,\ \ldots\ \alpha_{n+1} = \frac{A_1}{V},\ \frac{A_2}{V},\ \ldots\ \frac{A_{n+1}}{V}.$$

The element of area of the surface (as will be shown hereafter) is

$$dS = Vdu_1du_2\ldots du_n,$$

or by virtue of the above relations,

$$dS = (\alpha_1A_1 + \alpha_2A_2 + \ldots + \alpha_{n+1}A_{n+1})\prod_1^n du_i.$$

Suppose we have now a second surface

$$\Psi(y_1y_2\ldots y_{n+1}) = 0$$

connected with the first by the relations

$$y_i = f_i(x_1x_2\ldots x_{n+1}) \qquad i = 1,\ 2\ldots n+1.$$

Denote by $\beta_1,\ \beta_2,\ldots\beta_{n+1}$ the direction-cosines of the normal to this second surface, and by $B_1,\ B_2\ \ldots\ B_{n+1}$ what the determinants A_i of the first surface become for the second surface; then writing

$$U^2 = B_1^2 + B_2^2 + \ldots + B_{n+1}^2$$

we have
$$\beta_i = \frac{B_i}{U}$$
and for the element of area
$$d\Sigma = (\beta_1 B_1 + \beta_2 B_2 + \ . \ . \ . \ + \beta_{n+1} B_{n+1}) \prod_1^n du_i \, .$$

The ratio between the elements of area dS and $d\Sigma$ is
$$\frac{d\Sigma}{dS} = \frac{U}{V}.$$

Now V may obviously be written in the form
$$V = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n+1} \\ \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \frac{dx_3}{du_1} & \ldots & \frac{dx_{n+1}}{du_1} \\ \frac{dx_1}{du_2} & \frac{dx_2}{du_2} & \frac{dx_3}{du_2} & \ldots & \frac{dx_{n+1}}{du_2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{dx_1}{du_n} & \frac{dx_2}{du_n} & \frac{dx_3}{du_n} & \ldots & \frac{dx_{n+1}}{du_n} \end{vmatrix},$$
and similarly, denoting by $\left(\frac{dy_i}{du_k}\right)$ the quantity
$$\frac{dy_i}{dx_1}\frac{dx_1}{du_k} + \frac{dy_i}{dx_2}\frac{dx_2}{du_k} + \ . \ . \ . + \frac{dy_i}{dx_{n+1}}\frac{dx_{n+1}}{du_k},$$
we may write
$$U = \begin{vmatrix} \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_{n+1} \\ \left(\frac{dy_1}{du_1}\right) & \left(\frac{dy_2}{du_1}\right) & \left(\frac{dy_3}{du_1}\right) & \ldots & \left(\frac{dy_{n+1}}{du_1}\right) \\ \left(\frac{dy_1}{du_2}\right) & \left(\frac{dy_2}{du_2}\right) & \left(\frac{dy_3}{du_2}\right) & \ldots & \left(\frac{dy_{n+1}}{du_2}\right) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \left(\frac{dy_1}{du_n}\right) & \left(\frac{dy_2}{du_n}\right) & \left(\frac{dy_3}{du_n}\right) & \ldots & \left(\frac{dy_{n+1}}{du_n}\right) \end{vmatrix}$$

The direction-cosines of the lines of intersection of the surfaces u taken $n-1$ at a time with the surface Φ are
$$\frac{1}{\sqrt{E_{11}}}\frac{dx_1}{du_1}, \ \frac{1}{\sqrt{E_{11}}}\frac{dx_2}{du_1}, \ . \ . \ . \ . \ \frac{1}{\sqrt{E_{11}}}\frac{dx_{n+1}}{du_1},$$

and so for the remaining cases. At the common point of intersection of these lines, the normal (α) is at right angles to them all, and so

$$\alpha_1 \frac{dx_1}{du_i} + \alpha_2 \frac{dx_2}{du_i} + \ldots + \alpha_{n+1} \frac{dx_{n+1}}{du_i} = 0, \quad i = 1, 2 \ldots n.$$

Also $\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_{n+1}^2 = 1$. The determinant U may then be replaced by

$$U = \begin{vmatrix} \beta_1 & \beta_2 & \ldots & \beta_{n+1}, & 0 \\ \left(\frac{dy_1}{du_1}\right) & \left(\frac{dy_2}{du_1}\right) & \ldots & \left(\frac{dy_{n+1}}{du_1}\right), & \sum_1^{n+1} \alpha_i \frac{dx_i}{du_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \left(\frac{dy_1}{du_n}\right) & \left(\frac{dy_2}{du_n}\right) & \ldots & \left(\frac{dy_{n+1}}{du_n}\right), & \Sigma \alpha_i \frac{dx_i}{du_n} \\ \sum_1^{n+1} \alpha_i \frac{dy_1}{dx_i} & \Sigma \alpha_i \frac{dy_2}{dx_i} & \ldots & \Sigma \alpha_i \frac{dy_{n+1}}{dx_i}; & \Sigma \alpha_i^2 \end{vmatrix}$$

This is however the product of two determinants, viz.

$$U = \begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \ldots & \frac{dx_{n+1}}{du_1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \frac{dx_1}{du_n} & \frac{dx_2}{du_n} & \ldots & \frac{dx_{n+1}}{du_n} \\ 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{vmatrix} \begin{vmatrix} \beta_1 & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \ldots & \frac{dy_1}{dx_{n+1}} \\ \beta_2 & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \ldots & \frac{dy_2}{dx_{n+1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_{n+1} & \frac{dy_{n+1}}{dx_1} & \frac{dy_{n+1}}{dx_2} & \ldots & \frac{dy_{n+1}}{dx_{n+1}} \\ 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{vmatrix}.$$

The first of these is however equal to V, so we find for the ratio of the corresponding elements of area on the two surfaces,

$$\frac{d\Sigma}{dS} = \frac{U}{V} = \begin{vmatrix} \beta_1 & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \ldots & \frac{dy_1}{dx_{n+1}} \\ \beta_2 & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \ldots & \frac{dy_2}{dx_{n+1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \beta_{n+1} & \frac{dy_{n+1}}{dx_1} & \frac{dy_{n+1}}{dx_2} & \ldots & \frac{dy_{n+1}}{dx_{n+1}} \\ 0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n+1} \end{vmatrix},$$

or say for brevity,

$$\frac{d\Sigma}{dS} = \frac{U}{V} = K.$$

This is rather an interesting formula, and one which I have not been able to find given anywhere else; I actually obtained it first for three dimensions, but the generalization was obvious and attended with no difficulty, so for the purposes of another part of the paper I prefer to give it in this form at once. If we denote by x_1, x_2, x_3 the coördinates of a point on an ordinary surface, and by y_1, y_2, y_3 the coördinates of the corresponding point on a second surface, we have*

$$\frac{d\Sigma}{dS} = \frac{U}{V} = \begin{vmatrix} \beta_1 & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \frac{dy_1}{dx_3} \\ \beta_2 & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \frac{dy_2}{dx_3} \\ \beta_3 & \frac{dy_3}{dx_1} & \frac{dy_3}{dx_2} & \frac{dy_3}{dx_3} \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}, \text{ say } k.$$

Suppose the second surface to be a sphere of radius unity, whose radii are parallel to the normals at the corresponding point of the first surface. Let the first surface be given in the form

$$x_3 = \phi(x_1 x_2)$$

and denote by p and q the first differential coefficients of x_3 with respect to x_1 and x_2; also as usual write r, s, t for the second differential coefficients of x_3 with respect to x_1 and x_2.

The second surface is

$$y_1^2 + y_2^2 + y_3^2 = 1;$$

this is satisfied by writing

$$Y_1 = \frac{p}{Q}, \quad Y_2 = \frac{q}{Q}, \quad Y_3 = \frac{-1}{Q}$$

where $Q = \sqrt{1+p^2+q^2}$. The ratio between the corresponding elements of area on the two surfaces is now

$$\begin{vmatrix} \frac{p}{Q}, & \frac{(1+q^2)r - pqs}{Q^3}, & \frac{(1+q^2)s - pqt}{Q^3}, & 0 \\ \frac{q}{Q}, & \frac{(1+p^2)s - pqr}{Q^3}, & \frac{(1+p^2)t - pqs}{Q^3}, & 0 \\ \frac{-1}{Q}, & \frac{pr + qs}{Q^3}, & \frac{ps + qt}{Q^3}, & 0 \\ 0, & \frac{p}{Q}, & \frac{q}{Q}, & \frac{-1}{Q} \end{vmatrix}.$$

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* Some time after the above had gone to press I was informed that Neumann had stated this theorem, without proof, for space of three dimensions, in Volume IX of the *Mathematische Annalen.*—T. C.

This reduces to

$$\frac{-1}{Q^4}\begin{vmatrix} p, & (1+q^2)r-pqs, & (1+q^2)s-pqt \\ q, & (1+p^2)s-pqr, & (1+p^2)t-pqs \\ -1, & pr+qs, & ps+qt \end{vmatrix}.$$

Multiply the first row of the determinant by p, the second by q, and the third by -1, and add the first two rows to the last: we have then

$$\frac{1}{Q^3}\begin{vmatrix} (1+q^2)r-pqs, & (1+q^2)s-pqt \\ (1+p^2)s-pqr, & (1+p^2)t-pqs \end{vmatrix}.$$

Expanding this we have finally

$$k=\frac{rt-s^2}{(1+p^2+q^2)^2},$$

or calling R and R' the radii of curvature of the first surface,

$$k=\frac{1}{RR'},$$

the measure of curvature, as it should be. The reduction of the determinant in the case where the equation of the first surface is in the form

$$\Phi(x_1x_2x_3)=0$$

is a little more complicated, but attended with no particular difficulty. We of course must arrive at Gauss's expression for the measure of curvature in this case. The quantities $\alpha_1, \alpha_2, \alpha_3$ are the direction-cosines of the normal to the first surface at the point (x), and $\beta_1, \beta_2, \beta_3$ are the direction-cosines of the normal to the second surface, or sphere, at the corresponding point (y); then

$$\alpha_1,\ \alpha_2,\ \alpha_3=\beta_1,\ \beta_2,\ \beta_3=\frac{A_1}{V},\ \frac{A_2}{V},\ \frac{A_3}{V};$$

and of course

$$y_1,\ y_2,\ y_3=\frac{A_1}{V},\ \frac{A_2}{V},\ \frac{A_3}{V},$$

since the sphere is of radius unity. Consequently

$$\frac{d\Sigma}{dS}=k=\begin{vmatrix} \frac{A_1}{V}, & \frac{d}{dx_1}\cdot\frac{A_1}{V}, & \frac{d}{dx_2}\cdot\frac{A_1}{V}, & \frac{d}{dx_3}\cdot\frac{A_1}{V} \\ \frac{A_2}{V}, & \frac{d}{dx_1}\cdot\frac{A_2}{V}, & \frac{d}{dx_2}\cdot\frac{A_2}{V}, & \frac{d}{dx_3}\cdot\frac{A_2}{V} \\ \frac{A_3}{V}, & \frac{d}{dx_1}\cdot\frac{A_3}{V}, & \frac{d}{dx_2}\cdot\frac{A_3}{V}, & \frac{d}{dx_3}\cdot\frac{A_3}{V} \\ 0, & \frac{A_1}{V}, & \frac{A_2}{V}, & \frac{A_3}{V} \end{vmatrix}.$$

The reduction of this to the ordinary form is quite easy. Write

$$\frac{A_1}{V},\ \frac{A_2}{V},\ \frac{A_3}{V} = \cos\theta_1,\ \cos\theta_2,\ \cos\theta_3,$$

and we have

$$k = \begin{vmatrix} \cos\theta_1, & -\sin\theta_1\dfrac{d\theta_1}{dx_1}, & -\sin\theta_1\dfrac{d\theta_1}{dx_2}, & -\sin\theta_1\dfrac{d\theta_1}{dx_3} \\ \cos\theta_2, & -\sin\theta_2\dfrac{d\theta_2}{dx_1}, & -\sin\theta_2\dfrac{d\theta_2}{dx_2}, & -\sin\theta_2\dfrac{d\theta_2}{dx_3} \\ \cos\theta_3, & -\sin\theta_3\dfrac{d\theta_3}{dx_1}, & -\sin\theta_3\dfrac{d\theta_3}{dx_2}, & -\sin\theta_3\dfrac{d\theta_3}{dx_3} \\ 0, & \cos\theta_1, & \cos\theta_2, & \cos\theta_3, \end{vmatrix}.$$

Multiply the first row by $\cos\theta_1$, the second by $\cos\theta_2$, and the third by $\cos\theta_3$, and add the first and second rows to the third. The third row becomes now $1, 0, 0, 0$; interchanging the third and fourth rows, paying attention to all the signs, we have

$$k = \frac{\sin\theta_1\sin\theta_2}{\cos\theta_3}\begin{vmatrix} \dfrac{d\theta_1}{dx_1} & \dfrac{d\theta_1}{dx_2} & \dfrac{d\theta_1}{dx_3} \\ \dfrac{d\theta_2}{dx_1} & \dfrac{d\theta_2}{dx_2} & \dfrac{d\theta_2}{dx_3} \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{vmatrix}$$

Now

$$\frac{d}{dx} = \frac{du_1}{dx}\frac{d}{du_1} + \frac{du_2}{dx}\frac{d}{du_2};$$

changing in the determinant differential coefficients with respect to x into differential coefficients with respect to u_1 and u_2, we have after some easy reductions,

$$k = \frac{1}{\cos\theta_3}\left\{\frac{d\cos\theta_1}{du_1}\frac{d\cos\theta_2}{du_2} - \frac{d\cos\theta_1}{du_2}\frac{d\cos\theta_2}{du_1}\right\}\left\{\cos\theta_1\left(\frac{du_1}{dx_2}\frac{du_2}{dx_3} - \frac{du_1}{dx_3}\frac{du_2}{dx_2}\right)\right.$$
$$\left. + \cos\theta_2\left(\frac{du_1}{dx_3}\frac{du_2}{dx_1} - \frac{du_1}{dx_1}\frac{du_2}{dx_3}\right) + \cos\theta_3\left(\frac{du_1}{dx_1}\frac{du_2}{dx_2} - \frac{du_1}{dx_2}\frac{du_2}{dx_1}\right)\right\}.$$

Denote by E'_{11}, E'_{12}, E'_{22} the determinants

$$\begin{vmatrix} \dfrac{d^2x_1}{du_1^2} & \dfrac{d^2x_2}{du_1^2} & \dfrac{d^2x_3}{du_1^2} \\ \dfrac{dx_1}{du_1} & \dfrac{dx_2}{du_1} & \dfrac{dx_3}{du_1} \\ \dfrac{dx_1}{du_2} & \dfrac{dx_2}{du_2} & \dfrac{dx_3}{du_2} \end{vmatrix},\quad \begin{vmatrix} \dfrac{d^2x_1}{du_1du_2} & \dfrac{d^2x_2}{du_1du_2} & \dfrac{d^2x_3}{du_1du_2} \\ \dfrac{dx_1}{du_1} & \dfrac{dx_2}{du_1} & \dfrac{dx_3}{du_1} \\ \dfrac{dx_1}{du_2} & \dfrac{dx_2}{du_2} & \dfrac{dx_3}{du_2} \end{vmatrix},\quad \begin{vmatrix} \dfrac{d^2x_1}{du_2^2} & \dfrac{d^2x_2}{du_2^2} & \dfrac{d^2x_3}{du_2^2} \\ \dfrac{dx_1}{du_1} & \dfrac{dx_2}{du_1} & \dfrac{dx_3}{du_1} \\ \dfrac{dx_1}{du_2} & \dfrac{dx_2}{du_2} & \dfrac{dx_3}{du_2} \end{vmatrix}.$$

The quantities A_1, A_2, A_3 are the minors of these corresponding to the constituents of the first row in each. We find now readily

$$\frac{d\cos\theta_1}{du_1}=\frac{d}{du_1}\cdot\frac{A_1}{V}=-\frac{1}{V^3}\left[\left(E'_{11}E_{22}-E'_{12}E_{12}\right)\frac{dx_1}{du_1}+\left(E'_{12}E_{11}-E'_{11}E_{12}\right)\frac{dx_1}{du_2}\right]$$

$$\frac{d\cos\theta_1}{du_2}=\frac{d}{du_2}\cdot\frac{A_1}{V}=-\frac{1}{V^3}\left[\left(E'_{12}E_{22}-E'_{22}E_{12}\right)\frac{dx_1}{du_1}+\left(E'_{22}E_{11}-E'_{12}E_{12}\right)\frac{dx_1}{du_2}\right]$$

$$\frac{d\cos\theta_2}{du_1}=\frac{d}{du_1}\cdot\frac{A_2}{V}=-\frac{1}{V^3}\left[\left(E'_{11}E_{22}-E'_{12}E'_{12}\right)\frac{dx_2}{du_1}+\left(E'_{12}E_{11}-E'_{11}E_{12}\right)\frac{dx_2}{du_2}\right]$$

$$\frac{d\cos\theta_2}{du_2}=\frac{d}{du_2}\cdot\frac{A_2}{V}=-\frac{1}{V^3}\left[\left(E'_{12}E_{22}-E'_{22}E_{12}\right)\frac{dx_2}{du_1}+\left(E'_{22}E_{11}-E'_{12}E_{12}\right)\frac{dx_2}{du_2}\right]$$

Substituting these values in the first factor in brackets in the expression for k, we have, after easy reductions, for this factor the value

$$\frac{E'_{11}E'_{22}E'^2_{12}}{V^4}\left(\frac{dx_1}{du_1}\frac{dx_2}{du_2}-\frac{dx_1}{du_2}\frac{dx_2}{du_1}\right)=A_3\frac{E'_{11}E'_{22}-E'^2_{12}}{V^4}=\cos\theta_3\frac{E'_{11}E'_{22}-E'^2_{12}}{V^3}.$$

The remaining factor in brackets becomes obviously

$$\frac{A_1^2+A_2^2+A_3^2}{V^3}=\frac{1}{V}.$$

So finally collecting all the terms we have

$$k=\frac{E'_{11}E'_{22}-E'^2_{12}}{V^4},$$

or in the ordinary notation,

$$k=\frac{E'G'-F'^2}{(EG-F^2)^2}$$

the well-known form for the measure of curvature. We have, then, the theorem that if $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$ are the direction-cosines of the normal at any point x, y, z of a surface, the measure of curvature of the surface at this point is given by

$$k=\begin{vmatrix}\cos\theta_1, & \dfrac{d\cos\theta_1}{dx} & \dfrac{d\cos\theta_1}{dy} & \dfrac{d\cos\theta_1}{dz}\\ \cos\theta_2, & \dfrac{d\cos\theta_2}{dx} & \dfrac{d\cos\theta_2}{dy} & \dfrac{d\cos\theta_2}{dz}\\ \cos\theta_3, & \dfrac{d\cos\theta_3}{dx} & \dfrac{d\cos\theta_3}{dy} & \dfrac{d\cos\theta_3}{dz}\\ 0 & \cos\theta_1 & \cos\theta_2 & \cos\theta_3\end{vmatrix}$$

or again by

$$k = \frac{\sin\theta_1 \sin\theta_2}{\cos\theta_3} \begin{vmatrix} \frac{d\theta_1}{dx} & \frac{d\theta_1}{dy} & \frac{d\theta_1}{dz} \\ \frac{d\theta_2}{dx} & \frac{d\theta_2}{dy} & \frac{d\theta_2}{dz} \\ \cos\theta_1 & \cos\theta_2 & \cos\theta_3 \end{vmatrix}.$$

There are of course two other forms similar to this got by advancing the suffixes of θ. The bordered determinant whose value in the general case is K may in like manner be shown to be equal to

$$\frac{V'}{V^{n+2}},$$

where V' is the result of accenting all the letters in V just as has been done in the case of an ordinary surface and a unit sphere, so that $\frac{V'}{V^{n+2}}$ is the measure of curvature of a surface

$$\Phi(x_1 x_2 \ldots x_{n+1}) = 0$$

in M_{n+1}, the unit sphere being

$$Y_1^2 + Y_2^2 + \ldots + Y_{n+1}^2 = 1.$$

The correspondence between the surface $\Phi = 0$ and the n-dimensional unit sphere is of course just the same as in the ordinary case. The direction-cosines of the normal to Φ are

$$\frac{\Phi_1}{Q}, \quad \frac{\Phi_2}{Q}, \ldots \frac{\Phi_{n+1}}{Q}$$

the subscripts indicating differential coefficients and

$$Q = \sqrt{\sum_1^{n+1} \Phi_i^2}.$$

The required correspondence between the surface Φ and the sphere is produced by writing

$$Y_i = \frac{\Phi_i}{Q}.$$

Consider consecutive points on both surfaces: on the surface Φ these are given by

$$(x_1,\ x_2 \ldots x_{n+1}),\ \left(x_1 + \frac{dx_1}{du_i}\,du_i,\ x_2 + \frac{dx_2}{du_i}\,du_i \ldots\right)$$

and on the sphere the corresponding consecutive points are

$$(Y_k) \text{ and } \left(Y_k + \frac{dy_k}{du_i}du_i\right).$$

Consider two other points (ξ) and (η) lying in M_{n+1}, one near Φ and the other near the sphere. Denoting now by G and Γ the volumes of the small parallelopipedons formed by these two systems of points, we have as is well known (*vide* Beez, *Mathematische Annalen*, Vol. VI),

$$G = \begin{vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \dots & \xi_{n+1} - x_{n+1} \\ \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \dots & \frac{dx_{n+1}}{du_1} \\ \frac{dx_1}{du_2} & \frac{dx_2}{du_2} & \dots & \frac{dx_{n+1}}{du_2} \\ \dots & \dots & \dots & \dots \\ \frac{dx_1}{du_n} & \frac{dx_2}{du_n} & \dots & \frac{dx_{n+1}}{du_n} \end{vmatrix} \prod_1^n du_i,\ \Gamma = \begin{vmatrix} \eta_1 - y_1, & \eta_2 - y_2 & \dots & \eta_{n+1} - y_{n+1} \\ \frac{dy_1}{du_1} & \frac{dy_2}{du_1} & \dots & \frac{dy_{n+1}}{du_1} \\ \frac{dy_1}{du_2} & \frac{dy_2}{du_2} & \dots & \frac{dy_{n+1}}{du_2} \\ \dots & \dots & \dots & \dots \\ \frac{dy_1}{du_n} & \frac{dy_2}{du_n} & \dots & \frac{dy_{n+1}}{du_n} \end{vmatrix} \prod_1^n du_i$$

The minors of these determinants corresponding to $(\xi - x)$ in the one and $(\eta - y)$ in the other are obviously the quantities A and B of the earlier part of the paper; and of course

$$\Sigma A^2 = V^2, \qquad \Sigma B^2 = U^2,$$

and

$$\frac{A_i}{V} = \frac{\Phi_i}{Q}, \qquad \frac{B_i}{U} = Y_i.$$

Call the distance between the points (x) and (ξ) δ, then expand the above determinant and divide through by $V.\delta.\prod_1^n du_i$; this gives

$$\frac{G}{\delta.V.\Pi du} = \Sigma \frac{A_i}{V} \cdot \frac{\xi_i - x_i}{\delta}.$$

Now $\left(\frac{A_i}{V}\right)$ are the direction-cosines of the normal to Φ at the point (x), and $\left(\frac{\xi_i - x_i}{\delta}\right)$ are the direction-cosines of the straight line joining (x) and (ξ). Denoting by θ the angle between the normal and this line $(x\xi)$ we have

$$\cos\theta = \sum_1^{n+1} \frac{A_i}{V} \cdot \frac{\xi_i - x_i}{\delta},$$

and consequently

$$V \cdot \prod_1^n du_i = \frac{G}{\delta \cos\theta}.$$

G is the volume of the parallelopipedon, δ its slant height, and obviously $\delta\cos\theta$ its altitude; consequently $\frac{G}{\delta\cos\theta}$ is the base, or the element of the surface Φ is given by the quantity $V\cdot\prod_1^n du_i$.

This is the value of dS that has been employed. The proof that I have given is similar to one given by Beez in the *Math. Ann.;* but as I obtained my proof before reading Beez's article, and as the two proofs differ considerably in the methods of working them out, I let mine stand just as I obtained it. It is obvious that for the correspondence between the sphere and the surface Φ required in order to find the value of the measure of curvature, it is only necessary to write

$$Y_i = \frac{A}{V}$$

and so

$$\frac{dy_i}{du_k} = \frac{1}{V^2}\left\{V\frac{dA_i}{du_k} - A_i\frac{dV}{du_k}\right\}.$$

Consider the determinant B_1, *i. e.*

$$B_1 = \begin{vmatrix} \frac{dy_2}{du_1} & \frac{dy_2}{du_2} & \dots & \frac{dy_2}{du_n} \\ \frac{dy_3}{du_1} & \frac{dy_3}{du_2} & \dots & \frac{dy_3}{du_n} \\ \dots & \dots & \dots & \dots \\ \frac{dy_{n+1}}{du_1} & \frac{dy_{n+1}}{du_2} & \dots & \frac{dy_{n+1}}{du_n} \end{vmatrix}.$$

Substituting in this the value of each constituent as given by the above formula, we find after simple reductions,

$$B_1 = \frac{1}{V^{n+2}}\begin{vmatrix} V^2 & A_2 & A_3 & \dots & A_{n+1} \\ \frac{1}{2}\frac{dV^2}{du_1} & \frac{dA_2}{du_1} & \frac{dA_3}{du_1} & \dots & \frac{dA_{n+1}}{du_1} \\ \frac{1}{2}\frac{dV^2}{du_2} & \frac{dA_2}{du_2} & \frac{dA_3}{du_2} & \dots & \frac{dA_{n+1}}{du_2} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{1}{2}\frac{dV^2}{du_n} & \frac{dA_2}{du_n} & \frac{dA_3}{du_n} & \dots & \frac{dA_{n+1}}{du_n} \end{vmatrix}.$$

Now since
$$V^2 = \sum_1^{n+1} A_i^2,$$
$$V\frac{dV}{du_i} = A_1\frac{dA_1}{du_i} + A_2\frac{dA_2}{du_i} + \ldots + A_{n+1}\frac{dA_{n+1}}{du_i};$$

so multiplying the second column of the determinant by A_2, the third by A_3, etc., and subtracting the sum of the products from the first column, we have

$$B_1 = \frac{A_1}{V^{n+2}}\begin{vmatrix} A_1 & A_2 & \ldots & A_{n+1} \\ \frac{dA_1}{du_1} & \frac{dA_2}{du_1} & \ldots & \frac{dA_{n+1}}{du_1} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dA_1}{du_n} & \frac{dA_2}{du_n} & \ldots & \frac{dA_{n+1}}{du_n} \end{vmatrix}.$$

By definition of the quantities A_i we have identically

$$1 = \frac{1}{V^2}\begin{vmatrix} A_1 & A_2 & \ldots & A_{n+1} \\ \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \ldots & \frac{dx_{n+1}}{du_1} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dx_1}{du_n} & \frac{dx_2}{du_n} & \ldots & \frac{dx_{n+1}}{du_n} \end{vmatrix};$$

also

$$\sum_{i=1}^{i=n+1} A_i\frac{dx_i}{du_k} = 0, \quad k = 1, 2, 3 \ldots n;$$

differentiate this and write as above

$$\begin{aligned} E'_{ik} &= \left(A_1\frac{d^2x_1}{du_i du_k} + A_2\frac{d^2x_2}{du_i du_k} + \ldots\right) \\ &= -\left(\frac{dA_1}{du_i}\frac{dx_1}{du_k} + \frac{dA_2}{du_i}\frac{dx_2}{du_k} + \ldots\right). \end{aligned}$$

Now multiplying together the last two determinants, we find readily for the measure of curvature the value

$$k = \frac{U}{V^{n+2}},$$

where U is the determinant formed out of the accented letters E'_{ik} in the same manner that V is formed from E_{ik}.

It is not necessary to go into the proof that the determinant

$$\begin{vmatrix} \dfrac{A_1}{V} & \dfrac{d}{du_1}\dfrac{A_1}{V} & \cdots & \dfrac{d}{du_n}\dfrac{A_1}{V} \\ \dfrac{A_2}{V} & \dfrac{d}{du_1}\dfrac{A_2}{V} & \cdots & \dfrac{d}{du_n}\dfrac{A_2}{V} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{A_{n+1}}{V} & \dfrac{d}{du_1}\dfrac{A_{n+1}}{V} & \cdots & \dfrac{d}{du_n}\dfrac{A_{n+1}}{V} \\ 0 & \dfrac{A_1}{V} & \cdots & \dfrac{A_{n+1}}{V} \end{vmatrix}$$

reduces in this case, as in the simple one of $n=2$, to the ratio

$$\frac{U}{V^{n+2}}$$

and consequently that this bordered determinant gives the measure of curvature for the n-dimensional surface $\Phi=0$ in M_{n+1}.

In seeking the expression for the radii of curvature of the surface $\Phi=0$ at any point, it will be sufficient to consider the case of $n+1=4$.

Denote by $\xi_1, \xi_2, \xi_3, \xi_4$ the coördinates of any point on the normal to $\Phi=0$ at the point x_1, x_2, x_3, x_4; then

$$\xi_i = x_i + A_i\lambda, \qquad i=1,\ 2,\ 3,\ 4.$$

Suppose (ξ) to be the point where this normal meets the consecutive normal, then

$$dx_i + A_i d\lambda + \lambda dA_i = 0;$$

but

$$dx_i = a_1^{(i)}du_1 + a_2^{(i)}du_2 + a_3^{(i)}du_3,$$

and similarly we may write

$$dA_i = A_1^{(i)}du_1 + A_2^{(i)}du_2 + A_3^{(i)}du_3;$$

consequently

$$(a_1^{(i)} + \lambda A_1^{(i)})\,du_1 + \ldots + A_i d\lambda = 0, \qquad i=1,\ 2,\ 3,\ 4.$$

Eliminating du_1, du_2, du_3 and $d\lambda$ from these we obtain

$$\begin{vmatrix} a_1^{(1)}+\lambda A_1^{(1)}, & a_2^{(1)}+\lambda A_2^{(1)}, & a_3^{(1)}+\lambda A_3^{(1)}, & A_1 \\ a_1^{(2)}+\lambda A_1^{(2)}, & a_2^{(2)}+\lambda A_2^{(2)}, & a_3^{(2)}+\lambda A_3^{(2)}, & A_2 \\ a_1^{(3)}+\lambda A_1^{(3)}, & a_2^{(3)}+\lambda A_2^{(3)}, & a_3^{(3)}+\lambda A_3^{(3)}, & A_3 \\ a_1^{(4)}+\lambda A_1^{(4)}, & a_2^{(4)}+\lambda A_2^{(4)}, & a_3^{(4)}+\lambda A_3^{(4)}, & A_4 \end{vmatrix} = 0.$$

Calling ρ the radius of curvature, we have $\rho^2 = \Sigma(\xi_i - x_i)^2 = \lambda^2 V^2$, and consequently

$$\lambda = \frac{\rho}{V}.$$

Substituting this in the previous equations and making one simple reduction, the equation becomes

$$\begin{vmatrix} E_{11}V + \rho\Sigma A_1^{(i)}a_1^{(i)}, & E_{12}V + \rho\Sigma A_1^{(i)}a_2^{(i)}, & E_{13}V + \rho\Sigma A_1^{(i)}a_3^{(i)} \\ E_{21}V + \rho\Sigma A_2^{(i)}a_1^{(i)}, & E_{22}V + \rho\Sigma A_2^{(i)}a_2^{(i)}, & E_{23}V + \rho\Sigma A_2^{(i)}a_3^{(i)} \\ E_{31}V + \rho\Sigma A_3^{(i)}a_1^{(i)}, & E_{32}V + \rho\Sigma A_3^{(i)}a_2^{(i)}, & E_{33}V + \rho\Sigma A_3^{(i)}a_3^{(i)} \end{vmatrix} = 0.$$

Introduce here the accented letters E'_{jk}, viz.

$$E'_{jk} = A_1 b_{jk}^{(1)} + A_2 b_{jk}^{(2)} + A_3 b_{jk}^{(3)} + A_4 b_{jk}^{(4)}$$

in which

$$b_{kk}^{(i)} = \frac{d^2 x_i}{du_k^2} = \frac{da_k^{(i)}}{du_k}$$

$$b_{jk}^{(i)} = \frac{d^2 x_i}{du_j du_k} = \frac{da_j^{(i)}}{du_k} = \frac{da_k^{(i)}}{du_j}.$$

Differentiate the identity

$$\sum_{j=1}^{j=4} A_j a_1^{(j)} = 0$$

and we obtain

$$\sum_{j=1}^{j=4} a_1^{(j)} A_i^{(j)} = -\sum_{j=1}^{j=4} A_j b_{1i}^{(j)}$$

There are in all six equations of this kind; the quantities on the left hand side of each of these equations are the coefficients of ρ in the above cubic equation, while the quantities on the right are equal to

$$-E_{11},\ -E_{22},\ -E_{33},\ -E_{12},\ -E_{31},\ -E_{23}.$$

The cubic equation for the determination of ρ becomes now

$$\begin{vmatrix} E'_{11}\rho - E_{11}V, & E'_{12}\rho - E_{12}V, & E'_{13}\rho - E_{13}V \\ E'_{21}\rho - E_{21}V, & E'_{22}\rho - E_{22}V, & E'_{23}\rho - E_{23}V \\ E'_{31}\rho - E_{31}V, & E'_{32}\rho - E_{32}V, & E'_{33}\rho - E_{33}V \end{vmatrix} = 0.$$

The equation for the determination of the radii of curvature of $\Phi = 0$ in M_{n+1} is obviously

$$\begin{vmatrix} E'_{11}\rho - E_{11}V, & E'_{12}\rho - E_{12}V & \ldots & E'_{1n}\rho - E_{1n}V \\ E'_{21}\rho - E_{21}V, & E'_{22}\rho - E_{22}V & \ldots & E'_{2n}\rho - E_{2n}V \\ \cdots & \cdots & \cdots & \cdots \\ E'_{n1}\rho - E_{n1}V, & E'_{n2}\rho - E_{n2}V & \ldots & E'_{nn}\rho - E_{nn}V \end{vmatrix} = 0.$$

The coefficient of ρ^n in this is U, where as before

$$U^2 = |E'_{jk}|, \qquad V^2 = |E_{jk}|,$$

and the constant term is V^{n+2}. Consequently denoting by $\rho_1, \rho_2 \ldots \rho_n$ the roots of this equation, we have

$$\frac{1}{\rho_1 \rho_2 \ldots \rho_n} = \frac{U}{V^{n+2}},$$

the value of the measure of curvature.

The equation

$$U = V^{n+2}$$

is the differential equation of all surfaces developable upon the n-dimensional sphere. The result of equating the coefficient of ρ to zero, *i. e.*

$$\rho_1 + \rho_2 + \ldots + \rho_n = 0$$

corresponds to a class of surfaces similar to surfaces of minimum area.

Denoting by $\alpha_1, \alpha_2, \alpha_3$ the direction-cosines of a normal to the surface $\Phi(x_1 x_2 x_3) = 0$ and by $\beta_1, \beta_2, \beta_3$ the direction-cosines to the second surface at the corresponding point, write

$$\begin{aligned}
l_1, \quad l_2, \quad l_3 &= \frac{d\alpha_1}{du_1}, \quad \frac{d\alpha_2}{du_1}, \quad \frac{d\alpha_3}{du_1} \\
m_1, \quad m_2, \quad m_3 &= \frac{d\alpha_1}{du_2}, \quad \frac{d\alpha_2}{du_2}, \quad \frac{d\alpha_3}{du_2} \\
\lambda_1, \quad \lambda_2, \quad \lambda_3 &= \frac{d\beta_1}{du_1}, \quad \frac{d\beta_2}{du_1}, \quad \frac{d\beta_3}{du_1} \\
\mu_1, \quad \mu_2, \quad \mu_3 &= \frac{d\beta_1}{du_2}, \quad \frac{d\beta_2}{du_2}, \quad \frac{d\beta_3}{du_2}.
\end{aligned}$$

Then obviously

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = Vk,$$

and, calling K the measure of curvature of the second surface,

$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} = UK.$$

Take now the determinant

$$\begin{vmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & 0 & 0 \\ \mu_1 & \mu_2 & \mu_3 & 0 & 0 \\ \frac{dy_1}{du_1} & \frac{dy_2}{du_1} & \frac{dy_3}{du_1} & \frac{E'}{V}, & \frac{F'}{V} \\ \frac{dy_1}{du_2} & \frac{dy_2}{du_2} & \frac{dy_3}{du_2} & \frac{F'}{V}, & \frac{G'}{V} \end{vmatrix}.$$

This is obviously

$$= V^2 k \,.\, UK.$$

Substituting for $\frac{dy}{du}$, &c., their values

$$\frac{dy}{du} = \frac{dy}{dx_1}\frac{dx_1}{du} + \frac{dy}{dx_2}\frac{dx_2}{du} + \frac{dy}{dx_3}\frac{dx_3}{du}, \text{ \&c.}$$

this determinant is readily seen to be equal to the product

$$V \begin{vmatrix} \lambda_1 & \mu_1 & \beta_1 & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \frac{dy_1}{dx_3} \\ \lambda_2 & \mu_2 & \beta_2 & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \frac{dy_2}{dx_3} \\ \lambda_3 & \mu_3 & \beta_3 & \frac{dy_3}{dx_1} & \frac{dy_3}{dx_2} & \frac{dy_3}{dx_3} \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & 0 & l_1 & l_2 & l_3 \\ 0 & 0 & 0 & m_1 & m_2 & m_3 \end{vmatrix}.$$

This is obviously $= V^2 UkK$, so that no results of any consequence can be obtained by bordering the functional determinant

$$\frac{d\,(y_1 y_2 y_3)}{d\,(x_1 x_2 x_3)}$$

in this manner. Take the determinant

$$\begin{vmatrix} \frac{dx_1}{du_2} & \frac{dx_1}{du_1} & \frac{dy_1}{dx_1} & \frac{dy_1}{dx_2} & \frac{dy_1}{dx_3} \\ \frac{dx_2}{du_2} & \frac{dx_2}{du_1} & \frac{dy_2}{dx_1} & \frac{dy_2}{dx_2} & \frac{dy_2}{dx_3} \\ \frac{dx_3}{du_2} & \frac{dx_3}{du_1} & \frac{dy_3}{dx_1} & \frac{dy_3}{dx_2} & \frac{dy_3}{dx_3} \\ 0 & 0 & \frac{dx_1}{du_1} & \frac{dx_2}{du_1} & \frac{dx_3}{du_1} \\ 0 & 0 & \frac{dx_1}{du_2} & \frac{dx_2}{du_2} & \frac{dx_3}{du_2} \end{vmatrix}, = \Delta$$

and multiply this by the determinant giving V^2, bordered in a suitable manner. The product is readily found to give

$$
-V^2 \begin{vmatrix} \dfrac{dx_1}{du_1} & \dfrac{dx_2}{du_1} & \dfrac{dx_3}{du_1} \\ \dfrac{dx_1}{du_2} & \dfrac{dx_2}{du_2} & \dfrac{dx_3}{du_2} \\ \Sigma A \dfrac{dy_1}{dx} & \Sigma A \dfrac{dy_2}{dx} & \Sigma A \dfrac{dy_3}{dx} \end{vmatrix}
$$

$$
= -V^2 \left\{ A_1 \Sigma A_\iota \frac{dy_1}{dx_\iota} + A_2 \Sigma A_\iota \frac{dy_2}{dx_\iota} + A_3 \Sigma A_\iota \frac{dy_3}{dx_\iota} \right\}
$$

$$
= - V^2 \Sigma A_k \Sigma A_\iota \frac{dy_k}{dx_\iota}.
$$

Substituting for $\dfrac{dy}{dx}$ the value

$$
\frac{dy}{du_1}\frac{du_1}{dx} + \frac{dy}{du_2}\frac{du_2}{dx},
$$

and dropping the factor V^2, we have

$$
-\Delta = \Sigma A_\iota \frac{du_1}{dx_\iota} \Sigma A_\iota \frac{dy_\iota}{du_1} + \Sigma A_\iota \frac{du_2}{dx_\iota} \Sigma A_\iota \frac{dy_\iota}{du_2}.
$$

The first of the factors in this expression is proportional to the cosine of the angle between the normals to the surfaces $\Phi = 0$ and $u_1 = \text{const.}$ The second factor is proportional to the cosine of the angle between the normal to $\Phi = 0$ and the tangent to the curve $u_1 = \text{const.}$ traced on the second surface—say $\Psi = 0$. The first factor of the last term is proportional to the cosine of the angle between the normals to $\Phi = 0$ and $u_2 = \text{const.}$, and the last factor is proportional to the cosine of the angle between the normal to $\Phi = 0$ and the tangent to the curve $u_2 = \text{const.}$, $\Psi = 0$. Call these angles θ_1, ϕ_1, θ_2, ϕ_2, and write

$$
\Sigma \left(\frac{dy_\iota}{du_1} \right)^2 = \mathfrak{E}, \quad \Sigma \left(\frac{dy_\iota}{du_2} \right)^2 = \mathfrak{D}, \quad \Sigma \frac{dy_\iota}{du_1}\frac{dy_\iota}{du_2} = \mathfrak{F},
$$

$$
\Sigma \left(\frac{du_1}{dx_\iota} \right)^2 = L, \quad \Sigma \left(\frac{du_2}{dx_\iota} \right)^2 = M; \text{ we have then at once}
$$

$$
-\Delta = V^2 \{ \mathfrak{E} L \cos\theta_1 \cos\phi_1 + \mathfrak{D} M \cos\theta_2 \cos\phi_2 \}.
$$

If the surfaces $u = \text{const.}$ intersect $\Phi = 0$ orthogonally, we have $\cos\theta_1 = \cos\theta_2 = 0$ and so $\Delta = 0$. If $\cos\phi_1 = \cos\phi_2 = 0$, all the normals to the surface $\Phi = 0$ will be parallel to the normals at the corresponding points of $\Psi = 0$.

The determinant

$$\frac{d(y_1 y_2 \ldots y_{n+1})}{d(x_1 x_2 \ldots x_{n+1})}$$

bordered as above will lead to a similar result, viz.

$$-\Delta = V^2 \Sigma H_i L_i \cos\theta_i \cos\phi_i .$$

Denote by xyz a point on the first surface, then

$$\frac{1}{\nu}\begin{vmatrix} x & y & z \\ \frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} & \frac{dz}{dv} \end{vmatrix} = \cos\theta ,$$

θ denoting the angle between the radius vector from the origin to (xyz) and the normal, and $\nu = \sqrt{x^2+y^2+z^2}$. Denote by ρ and ψ the corresponding radius vector and angle on the second surface, then

$$\frac{1}{\rho}\begin{vmatrix} \xi & \eta & \zeta \\ \frac{d\xi}{du} & \frac{d\eta}{du} & \frac{d\zeta}{du} \\ \frac{d\xi}{dv} & \frac{d\eta}{dv} & \frac{d\zeta}{dv} \end{vmatrix} = \cos\psi .$$

Multiply together the two determinants

$$\begin{vmatrix} \xi & \frac{d\xi}{dx} & \frac{d\xi}{dy} & \frac{d\xi}{dz} \\ \eta & \frac{d\eta}{dx} & \frac{d\eta}{dy} & \frac{d\eta}{dz} \\ \zeta & \frac{d\zeta}{dx} & \frac{d\zeta}{dy} & \frac{d\zeta}{dz} \\ 0 & a & b & c \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du} \\ 0 & \frac{dx}{dv} & \frac{dy}{dv} & \frac{dz}{dv} \\ 0 & x & y & z \end{vmatrix}$$

a, b, c being the direction-cosines of the normal to the first surface at (xyz). The product of these two is readily found to be

$$= \nu\cos\theta \begin{vmatrix} \xi & \eta & \zeta \\ \frac{d\xi}{du} & \frac{d\eta}{du} & \frac{d\zeta}{du} \\ \frac{d\xi}{dv} & \frac{d\eta}{dv} & \frac{d\zeta}{dv} \end{vmatrix} = \nu\cos\theta \,.\, \rho\cos\psi$$

therefore

$$\cos\psi = \frac{1}{\rho}\begin{vmatrix} \xi & \frac{d\xi}{dx} & \frac{d\xi}{dy} & \frac{d\xi}{dz} \\ \eta & \frac{d\eta}{dx} & \frac{d\eta}{dy} & \frac{d\eta}{dz} \\ \zeta & \frac{d\zeta}{dx} & \frac{d\zeta}{dy} & \frac{d\zeta}{dz} \\ 0 & a & b & c \end{vmatrix}.$$

The ratio between the corresponding elements of area on two corresponding surfaces (xyz), $(\xi\eta\zeta)$ has been shown to be =

$$\begin{vmatrix} \alpha & \frac{d\xi}{dx} & \frac{d\xi}{dy} & \frac{d\xi}{dz} \\ \beta & \frac{d\eta}{dx} & \frac{d\eta}{dy} & \frac{d\eta}{dz} \\ \gamma & \frac{d\zeta}{dx} & \frac{d\zeta}{dy} & \frac{d\zeta}{dz} \\ 0 & a & b & c \end{vmatrix} = -\frac{U}{V}.$$

This can be given in a different form in the case when the surfaces

$$\xi = F_1(xyz)$$
$$\eta = F_2(xyz)$$
$$\zeta = F_3(xyz)$$

are orthogonal. The coördinates of a point on the second surface may be considered as given by the intersection of the three surfaces

$$F_1 = \text{const.},\quad F_2 = \text{const.},\quad F_3 = \text{const.}$$

The conditions for orthogonality of these surfaces are, writing ξ, η, ζ instead of F_1, F_2, F_3,

$$\frac{d\xi}{dx}\frac{d\eta}{dx} + \frac{d\xi}{dy}\frac{d\eta}{dy} + \frac{d\xi}{dz}\frac{d\eta}{dz} = 0$$
$$\frac{d\eta}{dx}\frac{d\zeta}{dx} + \frac{d\eta}{dy}\frac{d\zeta}{dy} + \frac{d\eta}{dz}\frac{d\zeta}{dz} = 0$$
$$\frac{d\zeta}{dx}\frac{d\xi}{dx} + \frac{d\zeta}{dy}\frac{d\xi}{dy} + \frac{d\zeta}{dz}\frac{d\xi}{dz} = 0.$$

Find the product of the last given determinant and the functional determinant

$$\frac{d(\xi\eta\zeta)}{d(xyz)}$$

bordered by 1, 0, 0, 0 for its first row and by the same for its first column.

Write

$$\left(\frac{d\xi}{dx}\right)^2+\left(\frac{d\xi}{dy}\right)^2+\left(\frac{d\xi}{dz}\right)^2=\Xi^2$$

$$\left(\frac{d\eta}{dx}\right)^2+\left(\frac{d\eta}{dy}\right)^2+\left(\frac{d\eta}{dz}\right)^2=H^2$$

$$\left(\frac{d\zeta}{dx}\right)^2+\left(\frac{d\zeta}{dy}\right)^2+\left(\frac{d\zeta}{dz}\right)^2=Z^2.$$

We have, then, for the required product

$$\begin{vmatrix} \alpha & \beta & \gamma & 0 \\ \Xi^2 & 0 & 0 & \Xi\cos\theta_1 \\ 0 & H^2 & 0 & H\cos\theta_2 \\ 0 & 0 & Z^2 & Z\cos\theta_3 \end{vmatrix}$$

where θ_1, θ_2, θ_3 are the angles which the normal to the original surface makes with the normals to the surfaces ξ, η, ζ at their common point of intersection. Expanding this we have

$$\Xi HZ\{HZ\alpha\cos\theta_1+Z\Xi\beta\cos\theta_2+\Xi H\gamma\cos\theta_3\}$$

and consequently

$$\frac{U}{V}=\frac{-\Xi HZ}{\frac{d(\xi\eta\zeta)}{d(xyz)}}\left\{HZ\alpha\cos\theta_1+Z\Xi\beta\cos\theta_2+\Xi H\gamma\cos\theta_3\right\}$$

or

$$=\frac{-\Xi^2H^2Z^2}{\frac{d(\xi\eta\zeta)}{d(xyz)}}\left\{\frac{\alpha\cos\theta_1}{\Xi}+\frac{\beta\cos\theta_2}{H}+\frac{\gamma\cos\theta_3}{Z}\right\}$$

If we denote for the moment by

$$\begin{matrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{matrix}$$

the direction-cosines of the normals to the surfaces ξ, η, ζ respectively, we have

$$\frac{d(\xi\eta\zeta)}{d(xyz)}=\Xi HZ\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}=\Xi HZ;$$

and consequently

$$\frac{d\Sigma}{dS}, = \frac{U}{V}, = -\Xi HZ\left\{\frac{\alpha\cos\theta_1}{\Xi} + \frac{\beta\cos\theta_2}{H} + \frac{\gamma\cos\theta_3}{Z}\right\}.$$

Of course a similar formula applies for a surface of n dimensions in M_{n+1}.

I will now work out a few simple properties of curves in a space of more than three dimensions. The results obtained are (with one exception which is referred to below) I believe new. For simplicity, at first consider only a space of four dimensions—M_4. A curve in M_4 in the ordinary acceptation of that term will depend only on one parameter—it may either be given by the values of its coördinates expressed as functions of this parameter, or as the intersection of three 3-dimensional surfaces, say

$$\Phi_1 = 0, \quad \Phi_2 = 0, \quad \Phi_3 = 0.$$

✓

Let u denote the parameter. First obtain the equation of the osculating flat space of three dimensions at any point of the curve. Four points on the curve will determine the osculating 3-flat. Take then four points on the curve, say A, B, C, D. The coördinates are

A	B	C	D
x_1	$x_1 + \Delta x_1$	$x_1 + 2\Delta x_1 + \Delta^2 x_1$	$x_1 + 3\Delta x_1 + 3\Delta^2 x_1 + \Delta^3 x_1$
x_2	$x_2 + \Delta x_2$	$x_2 + 2\Delta x_2 + \Delta^2 x_2$	$x_2 + 3\Delta x_2 + 3\Delta^2 x_2 + \Delta^3 x_2$
x_3	$x_3 + \Delta x_3$	$x_3 + 2\Delta x_3 + \Delta^2 x_3$	$x_3 + 3\Delta x_3 + 3\Delta^2 x_3 + \Delta^3 x_3$
x_4	$x_4 + \Delta x_4$	$x_4 + 2\Delta x_4 + \Delta^2 x_4$	$x_4 + 3\Delta x_4 + 3\Delta^2 x_4 + \Delta^3 x_4$

Now if $(\xi_1, \xi_2, \xi_3, \xi_4)$ is any point in the 3-flat passing through (x), its equation is

$$a_1(\xi_1 - x_1) + a_2(\xi_2 - x_2) + a_3(\xi_3 - x_3) + a_4(\xi_4 - x_4) = 0.$$

Substitute for (ξ) the coördinates of B and we have

$$a_1\Delta x_1 + a_2\Delta x_2 + a_3\Delta x_3 + a_4\Delta x_4 = 0.$$

In like manner substituting the coördinates of C and D in the place of (ξ), and using this last result to reduce the result of substituting the coördinates of C, and similarly reducing in the case of the point D, we have

$$\Sigma a\Delta^2 x = 0 \qquad \Sigma a\Delta^3 x = 0.$$

Eliminating (a) from the four equations,

$$\Sigma a(\xi - x) = 0 \qquad \Sigma a\Delta x = 0$$
$$\Sigma a\Delta^2 x = 0 \qquad \Sigma a\Delta^3 x = 0,$$

passing to the limit and dividing through by du^3, we have as the equation of the osculating 3-flat at the point (x) of the curve Φ

$$\begin{vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \xi_3 - x_3 & \xi_4 - x_4 \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \frac{dx_3}{du} & \frac{dx_4}{du} \\ \frac{d^2x_1}{du^2} & \frac{d^2x_2}{du^2} & \frac{d^2x_3}{du^2} & \frac{d^2x_4}{du^2} \\ \frac{d^3x_1}{du^3} & \frac{d^3x_2}{du^3} & \frac{d^3x_3}{du^3} & \frac{d^3x_4}{du^3} \end{vmatrix} = 0.$$

Of course in exactly the same manner we obtain

$$\begin{vmatrix} \xi_1 - x_1, & \xi_2 - x_2 & \dots & \xi_n - x_n \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \dots & \frac{dx_n}{du} \\ \dots & \dots & \dots & \dots \\ \frac{d^{n-1}x_1}{du^{n-1}} & \frac{d^{n-1}x_2}{du^{n-1}} & \dots & \frac{d^{n-1}x_n}{du^{n-1}} \end{vmatrix} = 0$$

as the osculating $(n-1)$-flat to a curve in M_n. In a similar manner we obtain for the equations of the osculating 2-flat (*i. e.* osculating plane) to the point (x) on the curve Φ in M_4

$$\begin{Vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \xi_3 - x_3 & \xi_4 - x_4 \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \frac{dx_3}{du} & \frac{dx_4}{du} \\ \frac{d^2x_1}{du^2} & \frac{d^2x_2}{du^2} & \frac{d^2x_3}{du^2} & \frac{d^2x_4}{du^2} \end{Vmatrix} = 0.$$

And generally for the osculating k-flat to a curve in M_n we find without difficulty

$$\begin{Vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \dots & \xi_n - x_n \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \dots & \frac{dx_n}{du} \\ \dots & \dots & \dots & \dots \\ \frac{d^kx_1}{du^k} & \frac{d^kx_2}{du^k} & \dots & \frac{d^kx_n}{du^k} \end{Vmatrix} = 0.$$

The equations of the tangent to the curve Φ at the point (x) are of course

$$\begin{Vmatrix} \xi_1 - x_1 & \xi_2 - x_2 & \xi_3 - x_3 & \xi_4 - x_4 \\ \frac{dx_1}{du} & \frac{dx_2}{du} & \frac{dx_3}{du} & \frac{dx_4}{du} \end{Vmatrix} = 0.$$

The direction-cosines of the perpendicular to the osculating 3-flat are

$$\alpha_1,\ \beta_1,\ \gamma_1,\ \eta_1 = \frac{1}{Q}\begin{vmatrix} \frac{dx_2}{du} & \frac{dx_3}{du} & \frac{dx_4}{du} \\ \frac{d^2x_2}{du^2} & \frac{d^2x_3}{du^2} & \frac{d^2x_4}{du^2} \\ \frac{d^3x_2}{du^3} & \frac{d^3x_3}{du^3} & \frac{d^3x_4}{du^3} \end{vmatrix}, \ \&c.$$

where of course

$$Q = \begin{vmatrix} \Sigma\left(\frac{dx}{du}\right)^2, & \Sigma\frac{dx}{du}\frac{d^2x}{du^2}, & \Sigma\frac{dx}{du}\frac{d^3x}{du^3} \\ \Sigma\frac{dx}{du}\frac{d^2x}{du^2}, & \Sigma\left(\frac{d^2x}{du^2}\right)^2, & \Sigma\frac{d^2x}{du^2}\frac{d^3x}{du^3} \\ \Sigma\frac{dx}{du}\frac{d^3x}{du^3}, & \Sigma\frac{d^2x}{du^2}\frac{d^3x}{du^3}, & \Sigma\left(\frac{d^3x}{du^3}\right)^2 \end{vmatrix}$$

Suppose we make s the independent variable, then

$$\Sigma\left(\frac{dx}{ds}\right)^2 = 1$$

and

$$\Sigma\frac{dx}{ds}\frac{d^2x}{ds^2} = \frac{1}{2}\,d\Sigma\left(\frac{dx}{ds}\right)^2 = 0,$$

$$\Sigma\frac{dx}{ds}\frac{d^3x}{ds^3} = d\Sigma\frac{dx}{ds}\frac{d^2x}{ds^2} - d\Sigma\left(\frac{dx}{ds}\right)^2 = -d\Sigma\left(\frac{dx}{ds}\right)^2,$$

$$\Sigma\frac{d^2x}{ds^2}\frac{d^3x}{ds^3} = \frac{1}{2}\,d\Sigma\left(\frac{d^2x}{ds^2}\right)^2.$$

Therefore

$$Q = \begin{vmatrix} 1 & 0 & -\Sigma\left(\frac{d^2x}{ds^2}\right)^2 \\ 0 & \Sigma\left(\frac{d^2x}{ds^2}\right)^2, & \frac{1}{2}\,d\Sigma\left(\frac{d^2x}{ds^2}\right)^2 \\ -\Sigma\left(\frac{d^2x}{ds^2}\right)^2, & \frac{1}{2}\,d\Sigma\left(\frac{d^2x}{ds^2}\right)^2, & \Sigma\left(\frac{d^3x}{ds^3}\right)^2 \end{vmatrix} = \Sigma\left(\frac{d^2x}{ds^2}\right)^2\Sigma\left(\frac{d^3x}{ds^3}\right)^2 - \Sigma\left(\frac{d^2x}{ds^2}\right)^3.$$

It would not be difficult to find the radius of curvature of the curve at any point by introducing the osculating 3-dimensional sphere and finding its intersection with the osculating 3-flat giving an osculating 2-dimensional sphere. It

is not worth while doing this however, as the reductions are rather long, and as the same thing has been done, in a different manner, by G. E. A. Brunel in Vol. XIX (page 42) of the *Mathematische Annalen.* M. Brunel's formula is, changing his notation slightly,

$$\frac{1}{R_p^2} = \frac{1}{N_1} \cdot \frac{N_{p+1} N_{p-1}}{N_p^2}$$

where

$$N_p = \left\| \begin{matrix} x_1' & x_2' & \dots\dots & x_n' \\ x_1'' & x_2'' & \dots\dots & x_n'' \\ \dots & \dots & \dots\dots & \dots \\ x_1^{(p)} & x_2^{(p)} & \dots\dots & x_n^{(p)} \end{matrix} \right\|^2$$

the accents denoting differential coefficients with respect to s.

Note on the Theory of Simultaneous Linear Differential or Difference Equations with Constant Coefficients.

By J. J. Sylvester.

This theory is virtually the same for differential as for finite-difference equations. The mere verbal part of the exposition being somewhat easier for the former of the two, I shall prefer in the first instance to deal with them, although the applications are more interesting when made to bear on the latter. Simple to the last degree as are the method of solution and the nature of the result, I do not find the one or the other set out, or even indicated, except in the most perfunctory manner, in the ordinary text-books. This brief notice, designed for the junior readers of the Journal, is intended to supply the lacuna.

Let $u_{j,k}$ denote a linear function, with constant coefficients, of u_k and of its first ε_j derivatives in respect to t.

Let
$$\begin{array}{c} u_{1,1}+u_{1,2}+\ldots+u_{1,i}=0 \\ u_{2,1}+u_{2,2}+\ldots+u_{2,i}=0 \\ \ldots\ldots\ldots\ldots\ldots \\ u_{i,1}+u_{i,2}+\ldots+u_{i,i}=0 \end{array}$$
be the system of differential equations proposed for integration.

Call
$$\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_i=\sigma.$$

The process of arriving at the reducing equation for any one of the variables is after the manner of the dialytic method of elimination, viz.:

Along with the first equation take each of its $(\sigma-\varepsilon_1)^{\text{th}}$ derivatives, with the second equation each of its $(\sigma-\varepsilon_2)^{\text{th}}$ derivatives, . . . and with the i^{th} equation each of its $(\sigma-\varepsilon_i)^{\text{th}}$ derivatives.

There will thus come into existence $(i+1)\sigma-(\varepsilon_1+\varepsilon_2\ldots+\varepsilon_i)$ *i.e.* $i(\sigma+1)-\sigma$ equations between the $i(\sigma+1)$ quantities
$$\begin{array}{c} \omega_1,\ \delta_t\omega_1,\ \ldots\ \delta_t^\sigma\omega_1 \\ \omega_2,\ \delta_t\omega_2,\ \ldots\ \delta_t^\sigma\omega_2 \\ \ldots\ldots\ldots\ldots \\ \omega_i,\ \delta_t\omega_i,\ \ldots\ \delta_t^\sigma\omega_i \end{array}$$

If we omit those which appear in any one of the lines above written, there will remain $(\sigma+1)(i-1)$ or $i(\sigma+1)-\sigma-1$ which might be eliminated between the $i(\sigma+1)-\sigma$ equations, and there would thus result an equation between the quantities contained in the omitted line. This elimination, it will presently be seen, there is no occasion to perform; the noticeable algebraical fact about it is, that supposing it were performed, the form of the equation resulting between ω_k, $\delta_t\omega_k$, ... $\delta_t^\sigma\omega_k$ is invariable, whichever of the numbers 1, 2, 3, ... i be the value assigned to k.

Let the order of the highest derivative of each ω be reduced by one unit below the highest order previously taken, then there will be $i\sigma-\sigma$ or $(i-1)\sigma$ equations connecting the $i\sigma$ quantities

$$\begin{array}{llll} \omega_1, & \delta_t\omega_1, & \ldots & \delta_t^{\sigma-1}\omega_1 \\ \omega_2, & \delta_t\omega_2, & \ldots & \delta_t^{\sigma-1}\omega_2 \\ \ldots & \ldots & \ldots & \ldots \\ \omega_i, & \delta_t\omega_i, & \ldots & \delta_t^{\sigma-1}\omega_i \end{array}$$

and accordingly, if we omit the σ quantities which appear in any one (say the first) of the above lines, the remaining $(i-1)\sigma$ quantities may each of them be expressed as linear functions of ω_1 and its $(\sigma-1)$ derivatives: but the elimination previously indicated would lead to a homogeneous linear equation between ω_1 and its σ derivatives, and if in that, each argument $\delta_t^\lambda\omega_1$ be replaced by h^λ and $\lambda_1, \lambda_2, \ldots \lambda_\sigma$ be the σ roots of the algebraical equation so formed, it follows from the ordinary theory for a single equation that ω_1 (provided the given equations, and consequently the resulting ones, be left in their general form) will be of the form ${}^1C_1e^{h_1t}+{}^1C_2e^{h_2t}+\ldots+{}^1C_\sigma e^{h_\sigma t}$, and consequently by virtue of the previous remark $\omega_2, \omega_3, \ldots \omega_k$ will be of the same form as ω_1 (but, of course, with different coefficients), that is to say, the σ roots $h_1, h_2, \ldots h_\sigma$ are the same for the equation in σ_k as for the equation in σ_1, so that the coefficients in the equation between ω_k and its σ derivatives are, as premised, independent of the value of k.

Finally, to determine the equation whose roots are $h_1, h_2, \ldots h_\sigma$, let ${}^1Ce^{ht}$, one of the terms in the general value, be taken as a particular value of ω_1, which with corresponding values of the other ω's will serve to satisfy the given equations; $\omega_2, \omega_3, \ldots \omega_i$ being each of them linear functions of ω and derivatives of ω, must be of the forms ${}^2Ce^{ht}$, ${}^3Ce^{ht}$, ... ${}^iCe^{ht}$, so that $\omega_1, \omega_2, \ldots \omega_i$ and the derivatives of each of them will contain the common factor e^{ht}, and by

substitution in the original equations we shall obtain a system of simultaneous algebraical equations leading to the equation

$$\begin{vmatrix} R_{1,1}, & R_{1,2} & \dots & R_{1,i} \\ R_{2,1}, & R_{2,2} & \dots & R_{2,i} \\ \dots & \dots & \dots & \dots \\ R_{i,1}, & R_{i,2} & \dots & R_{i,i} \end{vmatrix} = 0 \quad .$$

where in general $R_{p,q}$ is what $u_{p,q}$ becomes on writing h^{μ} in place of $\delta_t^{\mu}\omega_q$.

The above determinant of the i^{th} order will be of degree $\varepsilon_1 + \varepsilon_2 \dots + \varepsilon_i$, *i. e.* of the degree σ (for the general case) in h, and the roots of the equation will give the σ values $h_1, h_2, \dots h_\sigma$.

It follows, therefore, that the result of the hypothetical elimination in the first instance referred to will be a linear function of $\delta_t^{\sigma}\omega_k, \delta_t^{\sigma-1}\omega_k, \dots \delta_t\omega_k, \omega_k$ of which the coefficients will be identical with the coefficients of $h^{\sigma}, h^{\sigma-1}, \dots h, 1$ in the above determinant. Hence no matter now what special values may be attributed to the coefficients of the given equations, the result last obtained remains of *universal* validity—without excepting those cases in which the result of the hypothetical elimination would be such that the corresponding algebraical equation possess equal roots, although in those cases the form assumed in the course of the argument for the value of ω_1 (viz. a linear function of exponentials) ceases to hold good. Neither for the same reason need any exception be made for those cases where the number of terms in the equation to ω_k falls below σ on account of one or more of the leading coefficients in the result of the hypothetical elimination becoming zero: the degree to which h rises in the determinant will be in all cases the right degree, whether it reaches the extreme possible limit σ or falls below it.

The result obtained may be briefly summarized as follows.

If

$$\begin{aligned} (\phi_1\delta_t)x + (\phi_2\delta_t)y + \dots + (\phi_i\delta_t)z &= 0 \\ (\psi_1\delta_t)x + (\psi_2\delta_t)y + \dots + (\psi_i\delta_t)z &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ (\omega_1\delta_t)x + (\omega_2\delta_t)y + \dots + (\omega_i\delta_t)z &= 0 \end{aligned}$$

(each $\phi, \psi, \dots, \omega$ standing for a rational-integral functional form) then will

$$(R\delta_t)x = 0 \quad (R\delta_t)y = 0 \dots (R\delta_t)z = 0,$$

where $R(\delta_t)$ is the resultant in respect to $x, y, \dots z$ of what the above equations become when δ_t is treated as an ordinary algebraical quantity; under

which form the proposition (by virtue of Euler's method of multipliers) becomes so nearly intuitive as to abrogate all necessity for any other demonstration.*

To pass to the parallel and more important theory in finite differences, it is only necessary to interpret $u_{j,k}$ to signify a linear function, with constant coefficients, of $(\omega_k)_t$, $(\omega_k)_{t+1}$, . . . $(\omega_k)_{t+\varepsilon_j}$, where t is the integer independent variable, (*say* $(\omega_k)_t$ and its ε_j difference-augmentatives), and instead of taking the differential derivatives of any one of the given equations, to take the corresponding difference-augmentatives. Then by precisely the same reasoning as before we shall have

$$\omega_{t+\sigma} + B\omega_{t+\sigma-1} + \ldots + L\omega_t = 0,$$

B, C, . . . L being so taken as that $h^\sigma + Bh^{\sigma-1} + \ldots + L$ shall be the determinant represented by the same form of matrix expressed by R's as before, but where $R_{p,q}$ is obtained from $u_{p,q}$ by writing h^θ in lieu of any argument $\omega_t + \theta$ which occurs in it.

The simplest example that can be given is where $i = 2$, $\varepsilon_1 = \varepsilon_2 = 1$,

$$u_{1,1} = -\eta_{t+1} + a\eta_t, \qquad u_{1,2} = b\theta_t,$$
$$u_{2,1} = c\eta_t \qquad\qquad u_{2,2} = -\theta_{t+1} + d\theta_t;$$

this was the case which occurred in the article on the extension of Tchebycheff's theorem, in the last number of the Journal, leading to the equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

and to expressions for η_t, θ_t as linear functions of λ_1^t, λ_2^t.

It may also be remarked that this same case gives an instantaneous solution of the problem proposed and successfully treated by Babbage in his Calculus of Functions, more than half a century ago, and since revived in connection with the theory of substitutions (Serret, *Alg. Sup.* 4 ed. tom. 2, pp. 256–262). The problem is to find $\phi x = \dfrac{ax + a}{\beta x + b}$ so that $\phi^i x$, say $\dfrac{a_i x + a_i}{\beta_i x + b_i}$, shall equal x for a given value of i.

To find in general $\phi^i x$ it is only necessary to solve the difference equations

$$u_i = au_{i-1} + av_{i-1}$$
$$v_i = \beta u_{i-1} + bv_{i-1},$$

and then u_i, v_i will, if $u_0 = 1$, $v_0 = 0$, coincide with a_i, β_i, and if $u_0 = 0$, $v_0 = 1$ with α_i, b_i.

*I regret that this simple reflection did not present itself to my mind before the preceding investigation, the necessity for which it does away with, had been set up in print. It of course applies equally well to the analogous proposition for finite-difference equations (u_t, v_t, ... being substituted for x, y . . . ., and $1 + \Delta$ for δ_t). This last named proposition, limited to the case of equations of the first order, is the foundation-stone of my new theory of Matrices regarded as Quantities, *i. e.* as subject to every kind of functional operation which ordinary arithmetical or algebraical quantities are or can be subject to: but though so important and so easily established, I know not where it can be found explicitly stated.

Thus calling ρ_1, ρ_2 the two roots of

$$\begin{vmatrix} -\rho + a & \alpha \\ \beta & -\rho + b \end{vmatrix} = 0,$$

α_i will be of the form $C(\rho_1^i - \rho_2^i)$ and β_i of the same form except as to C, say $\Gamma(\rho_1^i - \rho_2^i)$. Also a_i, b_i will be of the forms $C_1\rho_1^i + C_2\rho_2^i$, $\Gamma_1\rho_1^i + \Gamma_2\rho_2^i$, where $C_1 + C_2 = 1$, $\Gamma_1 + \Gamma_2 = 1$, and the required condition will be fulfilled, provided only that $\rho_1^i = \rho_2^i$, or say

$$\rho_1 = K\left(\cos\frac{\lambda\pi}{i} + \sqrt{-1}\,\sin\frac{\lambda\pi}{i}\right)$$

$$\rho_2 = K\left(\cos\frac{\lambda\pi}{i} - \sqrt{-1}\,\sin\frac{\lambda\pi}{i}\right)$$

i. e. if $(a+b)^2 - 4(ab - \alpha\beta)\left(\cos\frac{\lambda\pi}{i}\right)^2 = 1$, λ having any integer value (which without loss of generality may be taken inferior to i) except zero.*

If $\lambda = 0$, the two roots of the equation in ρ become equal and the form of the solution changes into

$$u_i = (C_1 + C_2 i)\rho^i \qquad v_i = (C_1' + C_2' i)\rho^i.$$

When $u_0 = 1$ and $v_0 = 0$ then $u_1 = a$, $v_0 = \beta$,

$$C_1 = 1,\ C_1' = 0, \qquad C_2 = \frac{a}{\rho} - 1,\ C_2' = \frac{\beta}{\rho},$$

and when $u_0 = 0$, $v_0 = 1$, $u_1 = \alpha$, $v_0 = b$,

$$C_1 = \frac{\alpha}{\rho},\ C' = \frac{b}{\rho} - 1, \qquad C_2 = 0,\ C_2' = 1,$$

and $\phi^i x = \dfrac{[\rho + i(a - \rho)]\,x + i\,\alpha}{i\beta x + \rho + (b - \rho)\,i}$ which cannot be periodic for any value of i,

and when $i = \infty$ becomes $\dfrac{(a-\rho)\,x + \alpha}{\beta x + b - \rho} = \dfrac{a - \rho}{\beta} = \dfrac{\alpha}{b - \rho}$, *i. e.* $= \dfrac{a - b}{2\beta}$ or $\dfrac{2\alpha}{a - b}$,

so that $\phi^i x$ in this case continually converges to a constant limit.

I may add that $\phi^i x$ converges to a constant limit not merely when the roots ρ_1, ρ_2 of

$$\begin{vmatrix} a - \rho & \alpha \\ \beta & b - \rho \end{vmatrix}$$

are equal, but whenever they are real. For the general form of $\phi^i x$, it may easily be found, is

$$\frac{[(\rho_2 - a)\,\rho_1^i + (\rho_1 - a)\,\rho_2^i]\,x + \alpha\,(\rho_1^i - \rho_2^i)}{\beta\,(\rho_1^i - \rho_2^i)\,x + [(\rho_2 - b)\,\rho_1^i + (\rho_1 - b)\,\rho_2^i]}$$

*There will thus be $(i-1)$ values of λ which will each give a distinct admissible solution of the problem of periodicity, but of course only those values of λ which are relatively prime to i will give primitive solutions. If $i = i'\delta$ the effect of making $\lambda = \lambda'\delta$ will be to make $\phi^i x = x$ by virtue of its making $\phi^{i'} x = 0$.

which if $\rho_2 > \rho_1$ when $i = \infty$ becomes $\frac{(\rho_1 - a)\,x - a}{-\beta x + \rho_1 - b} = \frac{a - \rho_1}{\beta}$ or $\frac{a}{b - \rho_1}$ where ρ_1 signifies the smaller of the two roots ρ_1, ρ_2; or in other words when $a - b > 2\sqrt{a\beta}$, the limiting value to $\phi^i x$, when ϕx represents $\frac{ax + a}{\beta x + b}$, is $\frac{(a-b) + \sqrt{(a-b)^2 - 4a\beta}}{2\beta}$, with the understanding that the quantity under the radical sign is to be taken positive.

So, if

$$x_{i+1} : y_{i+1} : z_{i+1} = ax_i + by_i + cz_i : a'x_i + b'y_i + c'z_i : a''x_i + b''y_i + c''z_i,$$

when all the roots of the determinant

$$\begin{vmatrix} a - \lambda & b & c \\ a' & b' - \lambda & c' \\ a'' & b'' & c'' - \lambda \end{vmatrix}$$

are real, the point x_i, y_i, z_i, as i increases, will be found to approach indefinitely near to a fixed straight line; and if all the roots are equal, to a fixed point.

The condition of the system of ratios $x_i : y_i : z_i$ being periodic and having a period m is tantamount to the condition that the m^{th} power of the matrix

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}$$

shall be the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{matrix}$$

The complete solution of this problem, and of the more general one of extracting the m^{th} root of any unit-matrix (*i. e.* a matrix in which each element in the principal diagonal is unity, and the rest zero), which constitutes the ultimate generalization of Babbage's problem and is soluble by the same method, will probably appear in a memoir on matrices, in the forthcoming number of the Journal.

In general, for a matrix of the order ω, the number of m^{th} roots is m^{ω} and each of them is perfectly determinate. But when the matrix is a unit-matrix or a zero-matrix (the latter meaning one in which every element is zero) there are distinct genera and species of such roots, and every species contains its own appropriate number of arbitrary constants.

Alhazen's Problem.

Its Bibliography and an Extension of the Problem.

By Marcus Baker.

Problem. *From two points in the plane of a circle, to draw lines meeting at a point in the circumference and making equal angles with the tangent drawn at that point.*

This problem is known as Alhazen's, and has been studied by several mathematicians, besides Alhazen, from the time of Huyghens to the present. The earliest solutions are all geometrical constructions in which the points are determined by the intersections of a hyperbola with the given circle. Later, analytical solutions were given, and lastly trigonometrical solutions.

The following list of references and their accompanying notes contains a condensed history of the problem.

Alhazen. Opticae thesavrvs Alhazeni arabis libri septem nunc primùm editi . . . à Risnero. 4 p.l. 288 pp. fol. *Basileae per episcopios* MDLXXII.

Alhazen was an Arabian who lived in the 11th century of the Christian era, dying at Cairo in 1038. He wrote this treatise on optics, which was first published, as above stated, in 1572, under the editorship of Risner. The first published solution of the problem is contained in this book, pp. 144–148, Props. 34, 38 and 39. The solution is effected by the aid of a hyperbola intersecting a circle, and is excessively prolix and intricate.

Analyst (The). A journal of pure and applied mathematics. Edited and published by J. E. Hendricks, A. M. 8vo. *Des Moines, Iowa: Mills & Co.* 1877. Vol. IV, No. 3, pp. 124–125.

A general solution was proposed, but the solution of only a special case was published.

Barrow (*Rev.* Isaac, *D. D.*) Lectiones xviii Cantabrigiae in scholis publicis habitae in qvibvs opticorum phaenomenΩn genuinae rationes investigantur, ac exponuntur. 8vo. *Londini, typis Gulielmi Godbid,* 1669. Lect. ix.

See also WHEWELL's edition of Barrow's works. 8vo. *Cambridge, University Press,* 1860. Lectiones opticae, pp. 84–89.

Barrow refers to Alhazen's "horribly prolix" solution, and discusses some of its special cases as applied to optics.

HUTTON (*Dr.* Charles). Trigonometrical solution of Alhazen's problem.

[*In* LEYBOURN (Thomas). The mathematical questions proposed in the Ladies' Diary (etc.) 8vo. *London: J. Mawman,* 1817. Vol. I, pp. 167–168.]

A trigonometrical solution by trial and error.

HUYGHENS (Christian) and SLUSE (Francis Rene). Excerpta ex epistolis nonnullis (etc.)

[*In* PHILOSOPHICAL TRANSACTIONS of the Royal Society. 4to. *London,* 1673. Vol. VIII, pp. 6119–6126, 6140–6146.]

Several solutions are contained in this correspondence, all of them geometrical, and all except one being by the aid of an hyperbola. One solution, by Sluse, is by means of the intersection of a parabola with the given circle. One of the solutions by Huyghens is the most elegant the problem has ever received.

KAESTNER (Abraham Gotthelf). Problematis Alhazeni analysis trigonometrica.

[*In* NOVI COMMENTARII societatis regiae scientiarum GOTTINGENSIS tomus VII, 1776. 4to. *Gottingae, J. C. Dieterich,* 1777, pp. 92–141. 1 pl.]

A complete trigonometrical solution, with application to several numerical examples, and containing differential equations to facilitate the computation.

LADIES' DIARY. 16mo. *London,* 1727. *Not seen.*

The problem concretely stated was the prize problem in the Diary for 1727. Solved the following year by trial and error by I . . . T . . .

LEYBOURN (Thomas). Geometrical construction of Alhazen's problem.

[*In* LEYBOURN (Thomas). The mathematical questions proposed in the Ladies' Diary (etc.) 8vo. *London: J. Mawman,* 1817. Vol. I, pp. 168–169.]

This construction is arranged from old material. The proof is given clearly and concisely, and a few bibliographic indications are added.

L'HOSPITAL (Guillaume François Antoine de). Traité analytique de sections coniques. 4to. *Paris: Montalant,* 1720. Livre X, Ex. vii, pp. 389–395.

In Book X, on determinate sections, Alhazen's problem is selected as an example of a problem whose geometrical solution can be effected by the aid of a conic section. Two solutions are given, essentially the same as those by Huyghens and Sluse, but with improved methods of arrangement and proof.

MAYER (Tobias). In a collection of problems by Mayer there is said to be a solution of Alhazen's problem. *Not seen.*

PHILOSOPHICAL TRANSACTIONS. *See* HUYGHENS and WALES.

PRIESTLY (Joseph). History of optics, translated into German with notes by Simon Klügel. 2 vols. Leipzig, 1775–6. *Not seen.*

Said to contain information about Alhazen's problem.

RISNER (Frederic)—Editor. *See* ALHAZEN.

ROBINS (Benjamin). Mathematical tracts. 2 vols. 8vo. *London*, 1761. *See* Vol. II, pp. 262–264.

One of these tracts is a scathing review of Robert Smith's "compleat system of optics." Robins points out the complete omission of Alhazen's problem, and supplies a short and easy proof of the correctness of one of Sluse's solutions.

SEITZ (Enoch Beery). Solution of a problem.

[*In* SCHOOL (The) VISITOR, devoted to the study of mathematics and grammar. 8vo. *Ansonia, Ohio: John S. Royer*, 1881. Vol. II, No. 2, February, pp. 24–25.]

A complete algebraical solution by an equation of the eighth degree, with numerical application and roots found by Horner's method.

SIMSON (Robert). Said to have solved the problem, but no solution is contained in any of his works accessible to the author.

SLUSE (Francis Rene). *See* HUYGHENS and SLUSE.

T . . . (I . . .). *See* LADIES' DIARY.

WALES (William). On the resolution of adfected equations.

[*In* PHILOSOPHICAL TRANSACTIONS of the Royal Society. 4to. *London*, 1781. Vol. LXXI, part 1, Ex. vi, pp. 472–476.]

Alhazen's problem is selected as furnishing an example of an adfected quadratic equation whose solution is easily effected by the aid of the logarithmic tables and his method of using them.

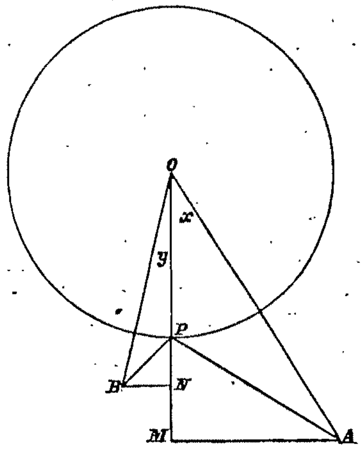
Alhazen's Problem extended to the Surface of a Sphere.

The solution of Alhazen's problem gives the minimum (and also maximum) path between two points and an intermediary circle, the points and circle being situated in the same plane, and we shall here give the solution of the same problem when the two points and circumference of the given circle are situated in the surface of a sphere.

To fix the ideas, consider the terrestrial spheroid to be a sphere, and let our given points be

A of which the latitude $= \phi_1$ and longitude λ_1,
and B " " " $= \phi_2$ " " λ_2.

The shortest distance between these points on the surface of the sphere is along the arc of the great circle joining them. Suppose this great circle track passes north of a given parallel of latitude ϕ; we wish to find the minimum path between A and B on the spherical surface which does *not* pass to the north of latitude ϕ.



The shortest path consists of two arcs of great circles drawn from A and B to a point P in latitude ϕ in such a manner as to make equal angles with the parallel ϕ at P. We proceed to determine this point P.

Referring to the annexed diagram, let us for a moment consider Alhazen's problem without the extension. Let $OA = a$, $OB = b$, and $OP = r$; then

$$\tan APM = \frac{a \sin x}{a \cos x - r} \text{ and } \tan BPN = \frac{b \sin y}{b \cos y - r},$$

and as these angles are equal we have

$$\frac{a \sin x\,(b \cos y - r)}{b \sin y\,(a \cos x - r)} = 1,$$

in which $x + y = \alpha$ a known angle. Replacing y by $\alpha - x$, we have a convenient formula for solving by approximation.

The solution for the extended problem is analogous to the foregoing. O is the pole of the sphere, A and B the given points, and P the point sought. OA, OB and OM are arcs of great circles, as are also PA and PB; and AM and BN are arcs of great circles perpendicular to OP. Let $AOM = x$, $BOM = y$, and $OP = 90° - \phi$.

Then
$$\sin AM = \sin AO \sin x,$$
$$\sin BN = \sin BO \sin y,$$

and
$$\tan OM = \tan AO \cos x,$$
$$\tan ON = \tan BO \cos y;$$

also $$\tan APM = \tan BPM \quad \text{or} \quad \frac{\tan AM}{\sin PM} = \frac{\tan BN}{\sin PN}$$

whence $$\frac{\tan^2 AM}{\tan^2 BN} = \frac{\sin^2 PM}{\sin^2 PN}. \tag{1}$$

Now
$$\frac{\tan^2 AM}{\tan^2 BN} = \frac{\sin^2 AM \cos^2 BN}{\cos^2 AM \sin^2 BN} = \frac{\sin^2 AO \sin^2 x\,(1 - \sin^2 BO \sin^2 y)}{(1 - \sin^2 AO \sin^2 x)\sin^2 BO \sin^2 y}; \tag{2}$$

and again, since $PN = ON - OP$ and $PM = OM - OP$,
$$\frac{\sin^2 PM}{\sin^2 PN} = \left\{ \frac{\sin OM \sin\varphi - \cos OM \cos\varphi}{\sin ON \sin\varphi - \cos ON \cos\varphi} \right\}^2 = \frac{\cos^2 OM}{\cos^2 ON} \left\{ \frac{\tan OM \sin\varphi - \cos\varphi}{\tan ON \sin\varphi - \cos\varphi} \right\}^2.$$

But
$$\frac{\cos^2 OM}{\cos^2 ON} = \frac{1 + \tan^2 ON}{1 + \tan^2 OM} = \frac{1 + \tan^2 BO \cos^2 y}{1 + \tan^2 AO \cos^2 x},$$

whence
$$\frac{\sin^2 PM}{\sin^2 PN} = \left\{ \frac{1 + \tan^2 BO \cos^2 y}{1 + \tan^2 AO \cos^2 x} \right\} \cdot \left\{ \frac{\tan AO \cos x \sin\varphi - \cos\varphi}{\tan BO \cos y \sin\varphi - \cos\varphi} \right\}^2. \tag{3}$$

Substituting (2) and (3) in (1), and remembering that
$$AO = 90^\circ - \phi_1 \quad \text{and} \quad BO = 90^\circ - \phi_2$$

we obtain
$$\frac{\cos^2\varphi_1 \sin^2 x\,(1 - \cos^2\varphi_2 \sin^2 y)}{(1 - \cos^2\varphi_1 \sin^2 x)\cos^2\varphi_2 \sin^2 y} = \left\{ \frac{1 + \operatorname{cotg}^2\varphi_2 \cos^2 y}{1 + \operatorname{cotg}^2\varphi_1 \cos^2 x} \right\} \cdot \left\{ \frac{\operatorname{cotg}\varphi_1 \sin\varphi \cos x - \cos\varphi}{\operatorname{cotg}\varphi_2 \sin\varphi \cos y - \cos\varphi} \right\}^2,$$

or
$$\frac{\cos^2\varphi_1 \sin^2 x\,(1 - \cos^2\varphi_2 \sin^2 y)\,(1 + \operatorname{cotg}^2\varphi_1 \cos^2 x)\,(\operatorname{cotg}\varphi_2 \cos y - \operatorname{cotg}\varphi)^2}{\cos^2\varphi_2 \sin^2 y\,(1 - \cos^2\varphi_1 \sin^2 x)\,(1 + \operatorname{cotg}^2\varphi_2 \cos^2 y)\,(\operatorname{cotg}\varphi_1 \cos x - \operatorname{cotg}\varphi)^2} = 1.$$

Now $x + y = \lambda_2 - \lambda_1 = \Delta\lambda$, whence eliminating x or y we have an equation which can be solved by approximation as in the case of a plane surface.

WASHINGTON, D. C., *October* 26, 1881.

On the Non-Euclidean Trigonometry.

By William E. Story.

In the "Sixth memoir upon quantics"* Professor Cayley has given a projective definition of geometrical quantity of one dimension (distance between two points and angle between two lines), which has been generalized by Dr. Klein in a paper "Ueber die sogenannte Nicht-Euklidische Geometrie."† Professor Cayley afterwards gave an example of the non-Euclidean trigonometry in the plane, obtaining the formulae for the special case in which the fundamental conic or "absolute" is a circle, and the constants of measurement have particular values.‡ I propose here to deduce the formulae for the general case of projective measurement in the plane with an arbitrary absolute and arbitrary constants. The application of the results to a space of two dimensions with constant curvature in a third dimension is evident. It is also evident from the nature of the projective measurement that the results are independent of the particular choice of coördinates, and therefore are not affected by the circumstance that a real point may have imaginary coördinates.

Following Professor Cayley's notation, I put a dash over any quantity measured projectively. The projective measure $\bar{a}$ of the distance between two points B and C, and the projective measure $\bar{A}$ of the angle between two lines b and c, are defined thus:

$$\bar{a} = k \ln \alpha\,, \quad \bar{A} = k' \ln \alpha'\,,$$

where α is the anharmonic ratio of B and C with respect to the intersections of the line BC with the absolute, and α' is the anharmonic ratio of b and c with

* Philosophical Transactions, Vol. 149, 1859. † Mathematische Annalen, Vol. IV, pp. 573–625.
‡ Mathematische Annalen, Vol. V, pp. 630–634.

respect to the tangents from the point bc to the absolute, and k and k' are two arbitrary constants corresponding to the arbitrary units of ordinary linear and angular measurement.

Let the equations of the absolute in point-coördinates x, y, z, and in tangential coördinates u, v, w, be

$$\Omega \equiv f(x, y, z) = 0 \quad \text{and} \quad \mho \equiv F(u, v, w) = 0,$$

respectively; and let

$$\Omega_{11} \equiv f(x_1, y_1, z_1), \qquad \Omega_{12} \equiv \frac{1}{2}\left[x_1 \frac{\partial \Omega_{22}}{\partial x_2} + y_1 \frac{\partial \Omega_{22}}{\partial y_2} + z_1 \frac{\partial \Omega_{22}}{\partial z_2}\right],$$

$$\mho_{11} \equiv F(u_1, v_1, w_1), \quad \mho_{12} \equiv \frac{1}{2}\left[u_1 \frac{\partial \mho_{22}}{\partial u_2} + v_1 \frac{\partial \mho_{22}}{\partial v_2} + w_1 \frac{\partial \mho_{22}}{\partial w_2}\right],$$

and similarly for other combinations of the suffices $1, 2, 3$.

Let A, B, C be the vertices of a triangle, a, b, c their sides, $\overline{A}, \overline{B}, \overline{C}$, $\overline{a}, \overline{b}, \overline{c}$ the projective measures of the angles and opposite sides, respectively; and let the coördinates of A, B, C, a, b, c, be x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3; u_1, v_1, w_1; u_2, v_2, w_2; u_3, v_3, w_3, respectively. Then, by the above definitions,

$$\overline{a} = k \ln \frac{\Omega_{23} + \sqrt{\Omega_{23}^2 - \Omega_{22}\Omega_{33}}}{\Omega_{23} - \sqrt{\Omega_{23}^2 - \Omega_{22}\Omega_{33}}},$$

$$\overline{A} = k' \ln \frac{\mho_{23} + \sqrt{\mho_{23}^2 - \mho_{22}\mho_{33}}}{\mho_{23} - \sqrt{\mho_{23}^2 - \mho_{22}\mho_{33}}},$$

which can be put into the forms

$$\overline{a} = 2ik \cos^{[-1]} \frac{\Omega_{23}}{\sqrt{\Omega_{22}\Omega_{33}}}, \quad \overline{A} = 2ik' \cos^{[-1]} \frac{\mho_{23}}{\sqrt{\mho_{22}\mho_{33}}},$$

i. e.

$$\cos\left(\frac{\overline{a}}{2ik}\right) = \frac{\Omega_{23}}{\sqrt{\Omega_{22}\Omega_{33}}}, \quad \cos\left(\frac{\overline{A}}{2ik'}\right) = \frac{\mho_{23}}{\sqrt{\mho_{22}\mho_{33}}}.$$

For convenience I take for the lines $x = 0$, $y = 0$, $z = 0$ respectively two conjugate polars through the point A and the polar of A with respect to Ω, so that

$$\Omega \equiv x^2 + y^2 + z^2, \text{ and hence } \mho \equiv u^2 + v^2 + w^2,$$

$$\Omega_{23} \equiv x_2x_3 + y_2y_3 + z_2z_3, \quad \mho_{23} \equiv u_2u_3 + v_2v_3 + w_2w_3,$$

$$x_1 : y_1 : z_1 = 0 : 0 : 1,$$

$$u_3 : v_3 : w_3 = y_2 : -x_2 : 0, \quad u_2 : v_2 : w_2 = y_3 : -x_3 : 0;$$

and, putting $x_2 = z_2 r_2 \cos\phi_2$, $y_2 = z_2 r_2 \sin\phi_2$, $x_3 = z_3 r_3 \cos\phi_3$, $y_3 = z_3 r_3 \sin\phi_3$, I find

$$\Omega_{11} = 1, \quad \Omega_{22} = z_2^2 (r_2^2 + 1), \quad \Omega_{33} = z_3^2 (r_3^2 + 1),$$

$$\Omega_{12} = z_2, \quad \Omega_{13} = z_3, \quad \Omega_{23} = z_2 z_3 [r_2 r_3 \cos(\phi_3 - \phi_2) + 1],$$

$$\mho_{22} = z_3^2 r_3^2, \quad \mho_{33} = z_2^2 r_2^2, \quad \mho_{23} = z_2 z_3 r_2 r_3 \cos(\phi_3 - \phi_2),$$

$$\cos\left(\frac{\bar{a}}{2ik}\right) = \pm \frac{1 + r_2 r_3 \cos(\varphi_3 - \varphi_2)}{\sqrt{(r_2^2+1)(r_3^2+1)}}, \quad \cos\left(\frac{\bar{A}}{2ik'}\right) = \pm \cos(\phi_3 - \phi_2),$$

$$\cos\left(\frac{\bar{b}}{2ik}\right) = \frac{\pm 1}{\sqrt{r_3^2+1}}, \quad \sin\left(\frac{\bar{b}}{2ik}\right) = \frac{\pm r_3}{\sqrt{r_3^2+1}},$$

$$\cos\left(\frac{\bar{c}}{2ik}\right) = \frac{\pm 1}{\sqrt{r_2^2+1}}, \quad \sin\left(\frac{\bar{c}}{2ik}\right) = \frac{\pm r_2}{\sqrt{r_2^2+1}},$$

$$\cos\left(\frac{\bar{a}}{2ik}\right) = \pm \cos\left(\frac{\bar{b}}{2ik}\right)\cos\left(\frac{\bar{c}}{2ik}\right) \pm \sin\left(\frac{\bar{b}}{2ik}\right)\sin\left(\frac{\bar{c}}{2ik}\right)\cos\left(\frac{\bar{A}}{2ik'}\right),$$

where each sign $\pm$ has to be so determined that the distance between two coincident points and the angle between two coincident lines shall be 0, or some multiple of the whole length of a straight line and whole angle about a point respectively, while it remains arbitrary in which of the two possible directions the measure is positive. If the points B and C coincide, $\bar{c} = \bar{b}$, $\bar{a} = 0$, $\bar{A} = 0$, hence, by the last formula,

$$1 = \pm \cos^2\left(\frac{\bar{b}}{2ik}\right) \pm \sin^2\left(\frac{\bar{b}}{2ik}\right),$$

therefore the upper sign is to be given to both terms, *i. e.*

$$\cos\left(\frac{\bar{a}}{2ik}\right) = \cos\left(\frac{\bar{b}}{2ik}\right)\cos\left(\frac{\bar{c}}{2ik}\right) + \sin\left(\frac{\bar{b}}{2ik}\right)\sin\left(\frac{\bar{c}}{2ik}\right)\cos\left(\frac{\bar{A}}{2ik'}\right),$$

corresponding to the formula of spherical trigonometry

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

from which, as is well known, all the other formulae of spherical trigonometry can be obtained, without the use of any other relations than those implied in the definitions of the trigonometric functions. Hence, in a uniformly curved space of two dimensions, with a projective measurement based upon any conic, whose constants of linear and angular measurement are k and k' respectively, *the trigonometrical formulae will be obtained from those of spherical trigonometry by replacing each side (or arc) by the corresponding side (or straight line) divided by*

$2ik$, *and each angle by the corresponding angle divided by* $2ik'$. Of course, what we call a straight line in such a curved space is a geodesic line, *i. e.* the shortest line between any two of its points measured as above.

If $k = \frac{1}{2}$, $k' = \frac{1}{2i}$,

then
$$\frac{\bar{a}}{2ik} = -i\bar{a}, \quad \frac{\bar{A}}{2ik'} = \bar{A},$$

$$\sin\left(\frac{\bar{a}}{2ik}\right) = -i\sinh\bar{a}, \quad \cos\left(\frac{\bar{a}}{2ik}\right) = \cosh\bar{a},$$

whence
$$\cos\bar{A} = -\frac{\cosh\bar{a} - \cosh\bar{b}\cosh\bar{c}}{\sinh\bar{b}\sinh\bar{c}},$$

the formula given by Professor Cayley, which, however, holds for any other fundamental conic, as well as for a circle.

BALTIMORE, *March* 28, 1882.

Note on Mechanical Involution.

By J. J. Sylvester.

Mechanical involution is the name invented by me to signify the relation between six lines in space, so situated that forces may be made to act along them whose statical sum is zero. The definition may be extended to comprise an indefinite number of lines, any six of which have this property.

I shall use $[p, q]$ for the present to denote the moment of a unit of force acting along the directed line p about the directed line q, taken positive or negative according as to a spectator looking in the given direction (or sense) of q, a force in the given direction (or sense) of p tends to produce a right-handed or a left-handed rotation, which tendency, by a property of our mental constitution, we know is not affected in kind by the lines p and q becoming interchanged—a fact which might also be anticipated with a high degree of probability from the circumstance that the unit-moment is measured by the product of the perpendicular distance from each other, of the two lines, multiplied by the sine of the angle between them, so that *each* factor of this product changes its sign when the relation or aspect of the two lines to each other is reversed. Hence it follows that $[p, q] = [q, p]$.

Three lines in a plane, it may be noticed, are in involution when they intersect in the same point, or, as a particular case, are parallel to each other.

Let a, b, c, d, e, f be any six lines in space, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ six forces capable of balancing when acting along the lines $1, 2, 3, 4, 5, 6$ supposed to be in involution.

Then by the equation of moments in regard to each of the first series of lines taken successively as axes of rotation, we must have

$$\begin{array}{l}
\lambda_1[1, a] + \lambda_2[2, a] + \lambda_3[3, a] + \lambda_4[4, a] + \lambda_5[5, a] + \lambda_6[6, a] = 0 \\
\lambda_1[1, b] + \cdots\cdots\cdots\cdots\cdots\cdots + \lambda_6[6, b] = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\lambda_1[1, f] + \lambda_2[2, f] + \lambda_3[3, f] + \lambda_4[4, f] + \lambda_5[5, f] + \lambda_6[6, f] = 0
\end{array}$$

and consequently the determinant

$$\begin{vmatrix} [1, a] & \cdot & \cdot & \cdot & \cdot & [6, a] \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ [1, f] & \cdot & \cdot & \cdot & \cdot & [6, f] \end{vmatrix} = 0.$$

Consequently we may find quantities $\mu_a, \mu_b, \mu_c, \mu_d, \mu_e, \mu_f$ such that

$$\mu_a[a, 1] + \mu_b[b, 1] + \mu_c[c, 1] + \mu_d[d, 1] + \mu_e[e, 1] + \mu_f[f, 1] = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mu_a[a, 6] + \mu_b[b, 6] + \mu_c[c, 6] + \mu_d[d, 6] + \mu_e[e, 6] + \mu_f[f, 6] = 0.$$

Thus it becomes evident by regarding $\mu_a, \mu_b, \mu_c, \mu_d, \mu_e, \mu_f$ as the magnitudes of forces acting along the lines a, b, c, d, e, f, that the equations of moments of a given set of forces about six lines which are in general independent, become linearly related when the six axes are in involution—a conclusion which springs also immediately from the consideration that the law of statical composition of directed lengths is the same whether they be regarded as representing forces or as representing the axes of couples. So much by way of introduction.

I now pass to the formation of the intrinsic equation of condition to be satisfied in the case of involution.

To obtain this, let the lines a, b, c, d, e, f be made identical with $1, 2, 3, 4, 5, 6$.

In each of these latter lines (say in i) let two points be taken at the distance $\frac{1}{l_i}$ apart, whose quadriplanar coördinates are respectively i_x, i_y, i_z, i_t, i'_x, i'_y, i'_z, i'_t, and let (i, j)—where j is another of the lines in involution—denote the determinant

$$\begin{vmatrix} i_x & i_y & i_z & i_t \\ i'_x & i'_y & i'_z & i'_t \\ j_x & j_y & j_z & j_t \\ j'_x & j'_y & j'_z & j'_t \end{vmatrix}.$$

This determinant will represent (enlarged six-fold) a tetrahedron, two of whose opposite edges are the lengths intercepted between the pairs of points on i, j respectively, and consequently $l_i l_j (i, j)$ will serve to represent (on the same scale) the quantities previously represented by $[i, j]$.

Hence the determinant of the sixth order above written becomes

$$\begin{vmatrix} 0 & l_1 l_2 (1, 2), & l_1 l_3 (1, 3), & l_1 l_4 (1, 4), & l_1 l_5 (1, 5), & l_1 l_6 (1, 6) \\ l_2 l_1 (2, 1), & 0, & l_2 l_3 (2, 3), & l_2 l_4 (2, 4), & l_2 l_5 (2, 5), & l_2 l_6 (2, 6) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ l_6 l_1 (6, 1), & l_6 l_2 (6, 2), & l_6 l_3 (6, 3), & l_6 l_4 (6, 4), & l_6 l_5 (6, 5), & 0 \end{vmatrix}$$

and this equated to zero gives the intrinsic condition of involution.

Imagining this equation to be formed, the terms in each line and also the terms in each column will have some common factor, removing which, by a two-fold scheme of division, all the quantities l will disappear, so that now regarding each of the pairs of points on the lines 1, 2, 3, 4, 5, 6 respectively as *any two non-coincident points whatever*, the intrinsic condition is represented by the evanescence of the following symmetrical invertebrate (*i. e.* zero-axial) compound determinant

$$\begin{vmatrix} 0 & (1, 2) & (1, 3) & (1, 4) & (1, 5) & (1, 6) \\ (2, 1) & 0 & (2, 3) & (2, 4) & (2, 5) & (2, 6) \\ (3, 1) & (3, 2) & 0 & (3, 4) & (3, 5) & (3, 6) \\ (4, 1) & (4, 2) & (4, 3) & 0 & (4, 5) & (4, 6) \\ (5, 1) & (5, 2) & (5, 3) & (5, 4) & 0 & (5, 6) \\ (6, 1) & (6, 2) & (6, 3) & (6, 4) & (6, 5) & 0 \end{vmatrix}$$

where each pair of numbers within a parenthesis represents a determinant of the fourth order.*

Just as the equations of moments of a system of forces about six lines in space are in general independent, but cease to be so if (and only if) these lines are in involution, so the equations of moments of a system of forces in a plane about three points are in general independent, and only cease to be so when the three points lie in a right line. Thus under the two-fold aspect of a system of force-directions and a system of axes of moments, six lines in involution in space are on the one hand the analogues of three force-directions in a plane in

* This determinant (which is sufficiently obvious, I have found since going to press) has been given by Prof. Cayley in his memoir on line-coördinates, Cam. Phil. Trans., 1861, which is avowedly based upon my constructions connected with the problem of Involution.

involution, *i. e.* meeting in a point, and on the other hand of three points (centres of moments) lying in a right line; and as *concurrence* is the polar correlative to *collineation* we ought to expect to find involution in space to be its own polar correlative; *i. e.* that the polar reciprocal of a system of lines in involution in respect to a general quadric should be another such system: and such is the fact: for, as I have shown in the *Comptes Rendus*, the necessary and sufficient condition of six lines being in involution is that they shall respectively intersect pairs of corresponding rays in two homographic pencils lying in two planes whose intersection contains the centres and two corresponding (coincident) rays of the two pencils—a condition which will not be affected by any polar transformation.

This leads to the remark that we may change the signification of the symbol (i, j) in the equation last indicated without destroying its validity as the condition of involution: viz. we may suppose two planes to be drawn through each line instead of two points being fixed upon it: and then if we understand by the determinant of two lines in space the determinant formed by the coefficients of the two pairs of equations which denote the lines, we may interpret (i, j) to mean the determinant of i, j and sum up the result obtained in the following proposition:

The determinants formed by six lines in involution, taken two and two together, are related in precisely the same manner as the squared distances from one another of six points in four-dimensional space.

The legitimacy of the second reading of (i, j) may be proved directly, as follows. For greater clearness let (i, j) when read with reference to pairs of planes through i and j, be called (I, J). Then

$$\begin{matrix} i_x & i_y & i_z & i_t \\ i'_x & i'_y & i'_z & i'_t \\ I_x & I_y & I_z & I_t \\ I'_x & I'_y & I'_z & I'_t \end{matrix}$$

will constitute an example of what in the Johns Hopkins University Circular for May, 1882,* I have called a *split matrix*, inasmuch as each of the first two

* Baltimore: John Murphy & Co.—It is interesting to notice (as there indicated) that the same theory of the split matrix here applied to mechanical involution has an important, although quite a different kind of bearing on the theory of algebraical involution. The two theories of involution have a considerable affinity to each other—groundforms and their coefficients in the equation of linear connection in the one theory, being regarded as the analogues of space-directions and the force-magnitudes acting along them in the other. (See J. H. U. Circular, June, 1882.) It was the sense of this connection which caused me to throw a retrospective glance on the theory of mechanical involution, abandoned by me since the remote date of the appearance of my papers on the subject in the *Comptes Rendus*. I ought to mention that I owe the idea of applying the split-matrix theory to the proof of the polar property of an involution-system, to a suggestion of Professor Cayley.

lines multiplied term for term by each of the latter two gives products whose sum is zero. Hence by virtue of the property of such a matrix, each complete minor of the upper pair will bear to the opposite complete minor in the lower pair the ratio of (i) to (I), where

$$(i)^2 = \begin{vmatrix} \Sigma i_x^2 & \Sigma i_x i'_x \\ \Sigma i_x i'_x & \Sigma i'^2_x \end{vmatrix} \text{ and } (I)^2 = \begin{vmatrix} \Sigma I_x^2 & \Sigma I_x I'_x \\ \Sigma I_x I'_x & \Sigma I'^2_x \end{vmatrix},$$

and of course the same conclusions apply *mut. mut.* when j, J take the place of i, I; from which it immediately follows that

$$(i, j) : (I, J) = (i)(j) : (I)(J).$$

Let now in the (i, j) determinant, which is equated to zero, each element in any θ^{th} column be multiplied by $\frac{I_\theta}{i_\theta}$, and then again each element in any θ^{th} row by the same; these multiplications will not affect the equality to zero of the determinant so modified, but the effect of the combined multiplications will be to change the element in the i^{th} row and j^{th} column, viz. (i, j), into $\frac{(I)(J)}{(i)(j)}(i, j)$, *i. e.* into (I, J). Thus it is proved that we may pass from the first reading of the (i, j) determinant to the second; and this in its turn serves to prove that if six lines are in involution their polars in respect to any quadric must also be in involution.

The theory of involution may of course be extended to a system of $\frac{n(n+1)}{2}$ lines in n-dimensional space.

Note on Determinants of Powers.

By O. H. Mitchell.

If $\alpha > \beta > \gamma > \delta$, then

$$\begin{vmatrix} a^\alpha & a^\beta & a^\gamma & a^\delta \\ b^\alpha & b^\beta & b^\gamma & b^\delta \\ c^\alpha & c^\beta & c^\gamma & c^\delta \\ d^\alpha & d^\beta & d^\gamma & d^\delta \end{vmatrix} \div \begin{vmatrix} a^3 & a^2 & a^1 & a^0 \\ b^3 & b^2 & b^1 & b^0 \\ c^3 & c^2 & c^1 & c^0 \\ d^3 & d^2 & d^1 & d^0 \end{vmatrix} = \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha^1 & \alpha^0 \\ \beta^3 & \beta^2 & \beta^1 & \beta^0 \\ \gamma^3 & \gamma^2 & \gamma^1 & \gamma^0 \\ \delta^3 & \delta^2 & \delta^1 & \delta^0 \end{vmatrix} \div \begin{vmatrix} 3^3 & 3^2 & 3^1 & 3^0 \\ 2^3 & 2^2 & 2^1 & 2^0 \\ 1^3 & 1^2 & 1^1 & 1^0 \\ 0^3 & 0^2 & 0^1 & 0^0 \end{vmatrix}$$

terms, *all positive* ($0^0 = 1$). In briefer notation, representing the first two determinants by their principal diagonals, and writing $\zeta^{\frac{1}{2}}(x, y, z, t)$ for $(x-y)(x-z)(x-t)(y-z)(y-t)(z-t)$,

$$\frac{(a^\alpha b^\beta c^\gamma d^\delta)}{(a^3 b^2 c^1 d^0)} = \frac{\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)}{\zeta^{\frac{1}{2}}(3, 2, 1, 0)} \text{ terms, } \textit{all positive};$$

and the like is true for determinants of any order. Calling the left-hand member Q_4, the subscript denoting the order of the determinants, Q_4 may be expressed as a linear function with positive coefficients of similar quotients, Q_3; and by successive applications of this formula Q_4 is expressible as $\Sigma\, a^p b^q c^r d^s$, where p, q, r, s are functions of the two sets of exponents, $\alpha, \beta, \gamma, \delta$ and $3, 2, 1, 0$, and of $(4-1)!$ variables whose limits in the summation are certain functions of $\alpha, \beta, \gamma, \delta$. And the like is true for determinants of any order.

For a proof of the preceding, consider the expression for Q_4 as given by Prof. Cayley, Salmon's Higher Algebra, p. 292, viz:

$$\frac{(a^\alpha b^\beta c^\gamma d^\delta)}{(a^3 b^2 c^1 d^0)} = \begin{vmatrix} H_{\alpha-3} & H_{\beta-3} & H_{\gamma-3} & H_{\delta-3} \\ H_{\alpha-2} & H_{\beta-2} & H_{\gamma-2} & H_{\delta-2} \\ H_{\alpha-1} & H_{\beta-1} & H_{\gamma-1} & H_{\delta-1} \\ H_\alpha & H_\beta & H_\gamma & H_\delta \end{vmatrix},$$

where $H_m =$ the sum of all possible powers and products of a, b, c, d of the m^{th} degree. If $\delta = 0$, the last row and the last column may be cast off, since $H_0 = 1$, and $H_{-m} = 0$. The left-hand member is divisible by $a^\delta b^\delta c^\delta d^\delta$; thus

$$\frac{(a^\alpha b^\beta c^\gamma d^\delta)}{(a^3 b^2 c^1 d^0)} = a^\delta b^\delta c^\delta d^\delta \, (H_{\alpha-\delta-3}\, H_{\beta-\delta-2}\, H_{\gamma-\delta-1}).$$

Expanding each element of the determinant on the right-hand side by means of the formula

$$H_m(a, b, c, d) = d^m + d^{m-1} H_1(a, b, c) + d^{m-2} H_2(a, b, c) + \text{etc.},$$

and resolving the determinant into the sum of similar determinants, we get

$$\frac{(a^\alpha b^\beta c^\gamma d^\delta)}{(a^3 b^2 c^1 d^0)} = a^\delta b^\delta c^\delta d^\delta \sum_{x=0}^{x=\alpha-\delta-1} \sum_{y=0}^{y=\beta-\delta-1} \sum_{z=0}^{z=\gamma-\delta-1} d^\theta \left| \begin{matrix} H_{x-2} & H_{y-2} & H_{z-2} \\ H_{x-1} & H_{y-1} & H_{z-1} \\ H_x & H_y & H_z \end{matrix} \right|_{(a,\, b,\, c)}$$

where the H's now involve only a, b, c, and $\theta = \alpha + \beta + \gamma + \delta - 3 - 4\delta - (x + y + z)$. A little consideration shows that the lower limit of x may be put $= \beta - \delta$ and the lower limit of y put $= \gamma - \delta$ without changing the value of the right-hand member; since the determinants obtained in the summation for lower values of x and y either vanish identically, having two columns the same, or destroy one another by twos, the members of each pair differing from each other only by an interchange of two columns. In fact, whatever x may be, the determinant obtained by putting $y = m$, $z = m$, vanishes identically, while that yielded by putting $y = m$, $z = n$, is destroyed by the one having $y = n$, $z = m$, the variable multiplier, d^θ, being the same for each. Of those determinants which remain, the one obtained by putting $x = l$ and either y or $z = l$ vanishes identically, and the one having $x = l$, $y = m$, $z = n$, is destroyed by that one whose $x = m$, $y = l$, $z = n$, if $\beta - \delta > l > \gamma - \delta - 1$, but by that one of which $x = n$, $y = m$, $z = l$, if $l < \gamma - \delta$; the variable coefficient, d^θ, does not interfere to prevent this mutual destruction of the quantities obtained in the summation, since $\theta = \alpha + \beta + \gamma - 3\delta - 3 - (x + y + z)$. For those determinants which now remain, $x > y > z$, and we have

$$\frac{(a^\alpha b^\beta c^\gamma d^\delta)}{(a^3 b^2 c^1 d^0)} = a^\delta b^\delta c^\delta d^\delta \sum_{x=\beta-\delta}^{x=\alpha-\delta-1} \sum_{y=\gamma-\delta}^{y=\beta-\delta-1} \sum_{z=0}^{z=\gamma-\delta-1} d^\theta \, \frac{(a^x b^y c^z)}{(a^2 b^1 c^0)}.$$

By successive applications of this formula we get,

$$\frac{(a^{\alpha_1}b^{\alpha_2}c^{\alpha_3}d^{\alpha_4})}{(a^3b^2c^1d^0)}=\sum_{x_1=\alpha_2-\alpha_4}^{x_1=\alpha_1-\alpha_4-1}\ \sum_{x_2=\alpha_3-\alpha_4}^{x_2=\alpha_2-\alpha_4-1}\ \sum_{x_3=0}^{x_3=\alpha_3-\alpha_4-1}\ \sum_{y_1=x_2-x_3}^{y_1=x_1-x_2-1}\ \sum_{y_2=0}^{y_2=x_2-x_3-1}\ \sum_{z_1=0}^{z_1=y_1-y_2-1} a^pb^qc^rd^s,$$

where
$$\begin{aligned}
p&=(\alpha_1+\alpha_2+\alpha_3+\alpha_4-3)-4\alpha_4-(x_1+x_2+x_3)+(\alpha_4),\\
q&=(x_1+x_2+x_3-2)-3x_3-(y_1+y_2)+(\alpha_4+x_3),\\
r&=(y_1+y_2-1)-2y_2-(z_1)+(\alpha_4+x_3+y_2),\\
s&=(z_1-0)-z_1-(0)+(\alpha_4+x_3+y_2+z_1).
\end{aligned}$$

The notation is slightly changed, and the values of p, q, r, s are written in a somewhat redundant form, in order to show more clearly the law for determinants of higher orders.

Since the number of terms in the H_m of n letters is expressed by the binomial coefficient C_{n-1}^{m+n-1}, the number of terms in the summation above (*i. e.* the sum of the coefficients, all of which are positive) is equal to the determinant

$$(C_3^{\alpha_1}C_3^{\alpha_2+1}C_3^{\alpha_3+2}C_3^{\alpha_4+3}),=(C_3^{\alpha_1}C_2^{\alpha_2}C_1^{\alpha_3}C_0^{\alpha_4}),=\frac{\zeta^{\frac{1}{2}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}{\zeta^{\frac{1}{2}}(3,2,1,0)},$$

obtained from the H-determinant. As an example,

$$\frac{(a^7b^3c^0)}{(a^2b^1c^0)}=\Sigma a^5b^2+\Sigma a^5bc+\Sigma a^4b^3+2\Sigma a^4b^2c+2\Sigma a^3b^3c+3\Sigma a^3b^2c^2,$$

which has 42, $=\dfrac{\zeta^{\frac{1}{2}}(7,3,0)}{\zeta^{\frac{1}{2}}(2,1,0)}$, terms.

If a_1, a_2, etc., be the n roots of $x^n-P_1x^{n-1}+P_2x^{n-2}-$ etc. $=0$, by substituting the roots and solving for P_r we get

$$P_r=\frac{(a_1^na_2^{n-1}\ldots a_r^{n-r+1}a_{r-1}^{n-r-1}\ldots a_{n-1}^1a_n^0)}{(a_1^{n-1}a_2^{n-2}\ldots a_r^{n-r}a_{r-1}^{n-r-1}\ldots a_{n-1}^1a_n^0)}.$$

Thus $\dfrac{(a^4b^3c^2d^0)}{(a^3b^2c^1d^0)}=\Sigma abc$, and since the number of terms in Σabc is C_3^4, we get the following expression for the value of the binomial coefficient, C_3^4, viz:

$$C_3^4=\frac{\zeta^{\frac{1}{2}}(4,3,2,0)}{\zeta^{\frac{1}{2}}(3,2,1,0)}.$$

And, in general, we have

$$C_r^n=\frac{\zeta^{\frac{1}{2}}(n,n-1,\ldots n-r+1,n-r-1,\ldots 2,1,0)}{\zeta^{\frac{1}{2}}(n-1,n-2,\ldots n-r,n-r-1,\ldots 2,1,0)},$$

the direct proof of which is of course very simple. Having proved that C_r^n has this value, it is easily shown, without reference to the roots of an equation, that P_r, *i. e.* $\Sigma a_1 a_2 \ldots a_r$, has the value obtained above. Thus, taking a special case for brevity, $\frac{(a^4b^3c^2d^0)}{(a^3b^2c^1d^0)}$ is seen to be a symmetric function, of the third degree, in which such a term as abc occurs *at least* once. Σabc is also such a function, in which abc occurs only once. Hence their equality is proved when it is shown that the number of terms in each is the same.

It may be remarked that if the last row of $(a^\alpha b^\beta c^\gamma)$ be subtracted from each of the preceding rows, and these rows be then divided respectively by $a-c$, $b-c$, and the second row be then subtracted from the first and the result divided by $a-b$, we get

$$\frac{(a^\alpha b^\beta c^\gamma)}{(a^2 b^1 c^0)} = \begin{vmatrix} H_{\alpha-2}(a,b,c), & H_{\beta-2}(a,b,c), & H_{\gamma-2}(a,b,c) \\ H_{\alpha-1}(b,c) & , \; H_{\beta-1}(b,c) & , \; H_{\gamma-1}(b,c) \\ H_\alpha(c) & , \; H_\beta(c) & , \; H_\gamma(c) \end{vmatrix}$$

and since $bH_{m-1}(b,c) + H_m(c) = H_m(b,c)$, and $aH_{m-1}(a,b,c) + H_m(b,c) = H_m(a,b,c)$, etc., the determinant on the right is at once reduced by combination of rows to the similar determinant already considered, in which each row involves every letter, and which is obtained in the text-books, as far as I have observed, only by an indirect process. The method obviously applies to determinants of any order. The following evident identity is used in the reduction:

$$H_m(a,c,d,e,\ldots z) - H_m(b,c,d,e,\ldots z) = (a-b)\,H_{m-1}(a,b,c,\ldots z).$$

The above process of reduction shows that the value of the H-determinant is the same for every distribution of the letters among the rows, subject to the conditions (1) that the letters of every row are contained in each of the preceding rows, and (2) that no row contains *less* than $\omega - r + 1$ letters, where ω is the order of the determinant, and r is the number of the row, counting from the top.

BALTIMORE, 20 *May*, 1882.

Determination of the Finite Quaternion Groups.

By W. I. Stringham,

Professor of Mathematics in the University of California.

If we apply to a set of quaternions the definition of a group as given in the Theory of Substitutions, then a *quaternion group* of the m^{th} order means a set of m quaternions (scalar unity always included) whose products and powers are also quaternions of the same set or group. The object of the present paper is to determine all the possible finite quaternion groups.

These groups, or rather their analogues in the ordinary Theory of Functions, have usually been interpreted geometrically as the linear transformations of the plane of complex variables in itself—*automorphic transformations.* The formulae for these linear transformations were first given by Professor Gordan in his paper "Ueber endliche Gruppen linearer Transformationen einer Veränderlichen."* The quaternion formulae, although they have their exact correspondents in Gordan's algebraic formulae, have, when interpreted geometrically, a more general character, in that they represent certain automorphic linear transformations of a three-dimensional infinite homoloidal space, or what is the same thing, of a three-fold extended sphere. The determination of these formulae might be made to depend upon Gordan's solution, but the following is a shorter and simpler solution than that of Gordan, and includes it as a special case.

Some General Formulae.

Let q, r represent two quaternions belonging to a finite group. In order that the group may be finite, the tensors of q, r, etc., must be unity, so that q and r may be written in the form

$$q = \cos\phi + \lambda \sin\phi,$$
$$r = \cos\psi + \mu \sin\psi,$$

* *Math. Ann.* Bd. XII, pp. 23–46. Compare also Cayley "On the finite Groups of linear Transformations of a Variable," *Math. Ann.* Bd. XVI, pp. 260–68, 439–40. The proof concerning the number and character of the possible groups was given by Prof. Klein, *Math. Ann.* Bd. IX, pp. 183–208: "Ueber binäre Formen mit linearen Transformationen in sich selbst."

ϕ, ψ being the arguments, λ, μ the axes of q, r respectively. According to definition qr is also a member of the group, and if we write

(i) $$qr = \cos\chi + \nu \sin\chi,$$

χ represents a new argument and ν a new axis of the group, and they are determined by the relations

(ii) $$\nu \sin\chi = \lambda \sin\phi \cos\psi + \mu \sin\psi \cos\phi + V\lambda\mu \sin\phi \sin\psi,$$

(iii) $$\cos\chi = \cos\phi \cos\psi + S\lambda\mu \sin\phi \sin\psi$$
$$= \cos^2\frac{f}{2} \cos(\phi + \psi) + \sin^2\frac{f}{2} \cos(\phi - \psi),$$

where f is the angle formed by λ with μ.

In particular, we have the well-known formulae

$$q^2 = \cos 2\phi + \lambda \sin 2\phi$$
$$\cdots\cdots\cdots\cdots$$

(iv) $$q^n = \cos n\phi + \lambda \sin n\phi.$$

In order that q may belong to a finite group, it is further necessary that some finite power of q should reproduce the *identity—i. e.* unity—so that, *e. g.*, $q^m = 1$; ϕ is then a rational multiple of $\eth$; m is called the period of q.

I. In the following discussion it is important to remember, that *when two quaternions of the same group have the same axis, they can always be written as powers of one and the same quaternion.*

Limits of the Periods.

Amongst the quaternions of the group there is one set—or one quaternion at least—whose argument is the smallest which the group contains. Let q be such a quaternion, and moreover such that its axis λ forms with some other axis μ of the group the smallest angle (the case $\lambda = \pm\mu$ excluded) which the axis of *any* quaternion of the group with the smallest argument can form with *any* other axis of the group. Under this hypothesis we have, if r be the quanternion whose axis is μ,

$$TV.\mu UVrqr \geqq TV\lambda\mu.$$

From the product rqr is deduced easily

$$V.\mu Vrqr = TVq.V\mu\lambda,$$

(v) $$V.\mu UVrqr = \frac{TVq}{TVrqr} \cdot V\mu\lambda.$$

In order that the condition $TV.\mu UVrqr \geqq TV\lambda\mu$ may be fulfilled, we must therefore have

(vi) $$V^2 q \geqq V^2 rqr\,, \text{ or } S^2 q \leqq S^2 qr^2\,,$$

i. e. $$\cos^2\phi \leqq \cos^2\theta\,,$$

where θ is the argument of rqr; ϕ, however, is by hypothesis the smallest argument which the group can possess, so that the only possible supposition is

$$\cos^2\phi = \cos^2\theta\,, \text{ or } Sqr^2 = \pm Sq\,.$$

Remembering that $S^2r - V^2r = 1$ (since $Tr = 1$) and that $r = Sr + Vr$, we have, if the sign before the second member be plus,

(vii) $$Sr\,.\,Sqr = 0\,,$$

but if minus,

$$S.rqr\,Vr = 0\,.$$

The latter of these conditions, however, would give

$$S\lambda\mu\,.\tan\phi = \tan\psi\,;$$

now as ψ by hypothesis cannot be smaller than ϕ, we should have

$$S\lambda\mu = \pm 1\,, \; i.e.\; \lambda = \pm\mu\,,$$

and r would be a power of q, a case we evidently do not need to consider. We have remaining therefore only the former condition $Sr\,.\,Sqr = 0$, which compels us to write either

$$Sr = 0\,, \text{ or } Sqr = 0\,.$$

The case $Sr = 0$ affords a possible solution, and determines for r the period 4.

But perhaps $Sqr = 0$ is also a solution. On the supposition that it is, write qr in the form

$$qr = q^{\frac{1}{2}}\, q^{\frac{1}{2}} r\, q^{\frac{1}{2}} q^{-\frac{1}{2}}\,.$$

This equation shows that the axes of qr and $q^{\frac{1}{2}} rq^{\frac{1}{2}}$ make equal angles with the axis of q. This angle cannot be smaller than the angle between λ and μ, and we have, if we rewrite the condition (vi) for this case,

(vi)′ $$S^2qr \geqq S^2r\,,$$

so that the condition $Sqr = 0$ demands also that $Sr = 0$; that is, the two conditions in question are equivalent, and r has in any case the period 4. We have therefore the following result:

II. *If q and r belong to a finite quaternion group, and if the argument of q is the smallest which the group contains, and if the angle between the axes of q and r is the smallest one existing between the axis of a quaternion with the smallest argument*

and any other axis of the group (the angle zero or $2\circleddash$ excepted), then $Sr = 0$ and r has of necessity the period 4.

One quaternion belonging to the group is

$$q' = rqr^{-1} = \cos\phi + \lambda' \sin\phi,$$

where $\lambda' = \mu\lambda\mu^{-1}$. If now we form the product $qq'q$—which must also be a quaternion of the group—we shall find, by the same process as that by which (v) was found, that

$$\text{(v)}' \qquad V.\lambda U Vqq'q = \frac{\sin\varphi}{TVqq'q}\cdot V\lambda\lambda',$$

and since ϕ is the smallest argument of the group,

$$\sin\phi \leqq TVqq'q,$$

from which and the preceding equation it follows, that

$$\text{(viii)} \qquad TV.\lambda.UVqq'q \leqq TV\lambda\lambda'.$$

This last inequality shows that the axis of $qq'q$, if it do not coincide with λ or λ', must lie between them and therefore coincide with μ, for μ is the only axis lying between these limits. We have therefore three possibilities:

$$qq'q = q'^x, \text{ or } qq'q = q^y, \text{ or } qq'q = \mu^z,$$

where x, y, z are integers. The case $qq'q = q'^x$, if we develop the equation $qq' = q'^x q^{-1}$ and operate with $V.\lambda$ on the result, gives

$$0 = V.\lambda\lambda' \operatorname{ctn}\phi + V.\lambda V\lambda\lambda',$$

$$\text{(ix)} \qquad 0 = V.\lambda\lambda' \operatorname{ctn}\phi - \lambda' - \lambda S\lambda\lambda';$$

and finally, by means of the operation $S.\lambda'$,

$$S^2\lambda\lambda' = 1,$$

which means simply—according to (I)—that q' is a power of q. The second case $qq'q = q^y$ and the case $qq'q = \pm\mu^2 = \pm 1$, give a similar result, since q' becomes respectively $q' = q^{y-2}$ and $q' = \pm q^{-2}$. We therefore conclude,

III. *That the quaternion $q\mu q\mu^{-1}q$, if it belong to a finite group, is either a power of q, or is equal to $\pm\mu$; provided μ, as compared with all the other axes of the group, forms with the axis of q the minimum angle.*

Groups which have a period greater than 10.

In order that the equation $qq'q = q^y$ (or $q' = q^{y-2}$) may be satisfied, we must have $\mu\lambda\mu^{-1} = \pm\lambda$, which demands further, that either

$$\text{(x)} \qquad V\lambda\mu = 0, \text{ or}$$

$$\text{(xi)} \qquad S\lambda\mu = 0,$$

that is, μ is either parallel or perpendicular to λ. If however $qq'q = \pm\, \mu$ should be satisfied, then

$$Sqq'q = \cos\phi \cos 2\phi + S\lambda\lambda' \sin\phi \sin 2\phi = 0,$$

$$\text{(xii)} \qquad S\lambda\lambda' = -\operatorname{ctn}\phi \operatorname{ctn} 2\phi.$$

The smallest value which ϕ can here assume is $\frac{\Theta}{6}$, and corresponding to this value we have

$$S\lambda\lambda' = -1, \qquad \lambda' = \lambda = \mu = q^3;$$

that is, r is a power of q, a case we do not consider. The condition $qq'q = \pm\, \mu$, therefore, is only possible when ϕ is greater than $\frac{\Theta}{6}$, *i. e.* when q has a period less than 12; and since it will be shown in the sequel (p. 353, VI) that the largest period of a finite group having more than a single axis must be an even period, it follows that the condition in question can exist only when the period of q does not exceed 10. We have, therefore, if the period of q be greater than 10, the single condition (xi), the condition (x) being irrelevant. Since of all the axes of the group μ has the greatest inclination to the axis of q, it follows from the results just obtained, that

IV. *If q has a period greater than* 10, *all the axes of the group, except that of q itself, lie in the plane—passing through the origin—to which the axis of q is perpendicular.*

It will next be shown, that

V. *Of a group containing a period greater than* 10, *all the quaternions which are not powers of q have either the period* 4 *or* 2.

If there be a quaternion s of the group, whose period is different from 4 or 2 and which is not a power of q, then there must be several such—with different axes, but with the same argument—which can be formed by means of the transformations qsq^{-1}, q^2sq^{-2}, . . . Let us suppose, therefore, that there are two such quaternions s, s' with the argument χ and the axes ν, ν'. Then by IV, either Vss' coincides with λ, or else it lies in the same plane with Vs and Vs'; that is, we must have either

$$S.\nu Vss' = S.\nu' Vss' = 0,$$

or

$$S.\nu\nu' Vss' = 0.$$

These two equations are equivalent respectively to

$$(S\nu\nu' - 1).\sin 2\chi = 0$$

and

$$V^2\nu\nu'.\sin^2\chi = 0.$$

Either equation would be satisfied by $S\nu\nu' = 1$, or what is the same thing, by $V\nu\nu' = 0$, in which case, however, ν and ν' would, contrary to hypothesis, coincide; hence χ must be so chosen as to make either $\sin 2\chi$ or $\sin\chi$ vanish, that is to say, χ must be an integral multiple either of $\eth$, or of $\frac{\eth}{2}$.—*Q.E.D.*

Cyclotomic (*Kreistheilungs*) *Groups.* If the condition $qq'q = q^\nu$, *i.e.* $V\lambda\mu = 0$, be fulfilled, all the axes of the group coincide with λ.

The group contains then only powers of a single quaternion q. If q be written in the form

$$q = \cos\frac{2\eth}{m} + \lambda\sin\frac{2\eth}{m},$$

we may assume as the generator of the group any power of q of the form q^a, where a is prime to m; the powers of q^a form a cyclotomic group of the m^{th} order. Since there are $\phi(m)$ numbers prime to m, there are $\phi(m)$ such cyclotomic groups composed of powers of q. Regarded as a whole, such groups are repetitions of each other written in different order.

If, besides the powers of q, there exist, as belonging to the group, only quaternions of the period 2, such quaternions are evidently all identical with each other and equal to -1. If this be combined with all the quaternions of a cyclotomic group of even order, say m, the resulting group will be a cyclotomic group of order $2m$, composed of the powers of $(-q)$. If, on the other hand, a cyclotomic group of even order be combined with -1, the result will be the same cyclotomic group again.

The Double-Pyramid Groups. If besides the m powers of q, m being even, there exists in the group a quaternion having the period 4, which we may represent by the vector μ—and μ is in this case also perpendicular to λ—then, by combining μ with all the powers of q, we can form m other quaternions

$$\mu q,\ \mu q^2,\ \ldots\ \mu q^m,$$

(or what is the same thing $q\mu$, $q^2\mu$, ... $q^m\mu$, since $\mu q^x = q^{-x}\mu$) which together with the m powers of q compose a group of the order $2m$. If, however, m be odd, then besides the $2m$ quaternions just mentioned, the group will contain $2m$ other quaternions

$$\mu^2 q,\ \mu^2 q^2,\ \ldots\ \mu^2 q^m \quad [\mu^2 = -1]$$
$$\mu^3 q,\ \mu^3 q^2,\ \ldots\ \mu^3 q^m \quad [\mu^3 = -\mu],$$

and the group will therefore be of the order $4m$. Hence the order of every quaternion group of this type is a multiple of 4.

Suppose $4m$ to be the order of a double-pyramid group. Then q has the period $2m$, and we have just seen that there are $\phi(2m)$ different powers of q, represented by q^a (where a is prime to m) any one of which will generate the cyclotomic group of the $2m^{\text{th}}$ order. We can use any one of these powers q^a as one of the generators of the double-pyramid group of the $2m^{\text{th}}$ order, μ being the other generator, and can thus obtain this group also in $\phi(2m)$ different forms.

Groups containing no Period greater than 10.

The Period 10. If q has a period not exceeding 10, we have then, for the determination of the angle between λ and μ, the condition (xii), p. 349, viz:

(xii) $$S.(\lambda\mu)^2 = S\lambda\lambda' = -\operatorname{ctn}\phi \operatorname{ctn} 2\phi.$$

For the period 10, therefore, we have

$$S\lambda\lambda' = -\cos 2f = -\operatorname{ctn}\frac{\partial}{5}\cdot\operatorname{ctn}\frac{2\partial}{5};$$

if $2f$ denote the angle which λ makes with λ', *i. e.* if f be the angle between λ and μ. Hence

$$\cos 2f = \frac{1}{\sqrt{5}},\quad \cos f = \sqrt{\frac{e}{\sqrt{5}}} = -S\lambda\mu$$

where $e = \frac{1+\sqrt{5}}{2}$. If we write $\mu = i$ and make the plane of (λ, μ) that of (i, k), we shall find the values of λ and q to be

$$\lambda = i\cos f + k\sin f = \frac{i - e'k}{\sqrt{1+e'^2}},$$

$$q = \cos\frac{\partial}{5} + \lambda\sin\frac{\partial}{5} = \frac{e + i - e'k}{2},$$

where $e' = \frac{1-\sqrt{5}}{2}$. By combining q and i and their powers and products in all possible ways, we shall generate a closed group consisting of 120 different quaternions:—the *Double-Ikosahedron Group.* q and i are called the generators of the group. (The quaternions of this and the following groups are written out in full in the accompanying tables $I_m - Q_{120}$, pp. 354, 355.)

Having chosen $\mu = i$, we might then have written, instead of

$$\lambda = i\cos f + k\sin f,$$

this value:

$$\lambda = i\cos f + j\sin f.$$

The latter assumption would evidently—on account of the symmetry of the i, j, k, in their relations to each other—result in a group similar to the

one determined by the former value of λ. It will be found that the two groups differ only by an interchange of e and e', and it is also easy to show that the former group becomes the latter by applying to it the transformation $(1-k)(\ )(1-k)^{-1}$; that is, by writing, in the former group, for each quaternion q the quaternion $(1-k).q.(1-k)^{-1}$. Two such groups are called similar and isomorphic*.

The Period 8. If q has the period 8, then $\phi = \frac{\pi}{4}$ and

$$S\lambda\lambda' = -\cos 2f = -\operatorname{ctn}\frac{\pi}{4}\operatorname{ctn}\frac{\pi}{2} = 0,$$

$$f = \frac{\pi}{4}.$$

Writing $\lambda = i$, we have

$$q = \frac{1+i}{\sqrt{2}},$$

and since μ forms with λ an angle of 45°, we may assume

$$\mu = \frac{i+j}{\sqrt{2}}.$$

q and μ will then be the generators of a closed group of 48 quaternions:—the *Double-Oktahedron Group.* The choice $\mu = \frac{i+k}{\sqrt{2}}$ could evidently equally well have been made; it turns out however that q and $\frac{i+k}{\sqrt{2}}$ generate the same group as q and $\frac{i+j}{\sqrt{2}}$.

The Period 6. If 6 be the period of q, then $\phi = \frac{\pi}{3}$ and

$$S\lambda\lambda' = -\cos 2f = -\operatorname{ctn}\frac{\pi}{3}\operatorname{ctn}\frac{2\pi}{3} = \frac{1}{3},$$

$$\cos f = \frac{1}{\sqrt{3}}.$$

Here, if we write $\mu = i$ and determine the plane (λ, μ) in such a way that it bisects the angle between j and k, we shall have for the values of λ and q

$$\lambda = \frac{j+k}{\sqrt{2}}\sin f + i\cos f = \frac{i+j+k}{\sqrt{3}},$$

$$q = \frac{1+i+j+k}{2}.$$

* C. Jordan, Traité des substitutions et des equations algebriques, pp. 23, 50.

q and i are the generators of a group of the 24th order:—the *Double-Tetrahedron Group.*

The Period 4. If q have the period 4, then $q = \lambda$. If the condition (xii) be applied to this case, we find $Sqq'q = 0 = S\lambda\lambda'$, $f = \frac{\partial}{4}$, and thus obtain the same value of f which gave rise to the double-oktahedron group; and we should in fact obtain this same group for the above value of f in the present case. This results from the fact that the product $\lambda\mu$ would be a quaternion having 8 for its period. We may therefore throw aside the condition (xii) and applying (xi)—*i. e.* $S\lambda\mu = 0$—write

$$\lambda = i, \quad \mu = j,$$

which two quaternions are the generators of the well-known group of the 8th order, viz. ± 1, $\pm i$, $\pm j$, $\pm k$. This is a special example of the double-pyramid type.

The Periods 11, 9, 7, 5, 3. In order to prove that the above enumeration of the finite quaternion groups is exhaustive, it will only be necessary to show,

VI. *That the greatest period contained in any finite quaternion group, which is not a cyclotomic group, is even.*

The smallest argument which can belong to an odd period m is $\frac{2\partial}{m}$. If such an argument however belong to a finite group which possesses more than a single axis, it cannot be the smallest argument in the group; for if q be the quaternion whose argument is $\frac{2\partial}{m}$, m being odd, and if μ be as before the axis most inclined to that of q, then $q m^3 q^{\frac{m-1}{2}} = -q^{\frac{m+1}{2}}$ belongs to the group, and since

$$\begin{aligned} -q^{\frac{m+1}{2}} &= -\cos\frac{m+1}{2}\partial - \lambda\sin\frac{m+1}{2}\partial \\ &= \cos\left(2m\partial + \frac{\partial}{m}\right) + \lambda\sin\left(2m\partial + \frac{\partial}{m}\right) \\ &= \cos\frac{\partial}{m} + \lambda\sin\frac{\partial}{m}, \end{aligned}$$

therefore the group contains the argument $\frac{\partial}{m}$, which is half of $\frac{2\partial}{m}$, and thus the proposition is proved. With this demonstration ends the possibility of forming any other class of finite quaternion groups than those above enumerated.

Tables of the Five Different Types of Groups.

The explicit formulae which constitute the five different types of groups are contained in the following tables. $N=$ order of the group.

I_m.

The Cyclotomic Groups.

These have the form

$$q^a,\ q^{2a},\ \ldots\ q^{ma};\quad N=m,\ a \text{ prime to } m;$$

where q is any quaternion having the period m.

J_{4n}.

1. *The Double-Pyramid Groups.*

These have the form

$$\begin{array}{l} q^a,\ q^{2a},\ \ldots\ q^{2na}, \\ \mu q^a,\ \mu q^{2a},\ \ldots\ \mu q^{2na};\quad N=4n,\ a \text{ prime to } 2n; \end{array}$$

where q is any quaternion having the period $2n$, and where μ is a vector lying in the plane perpendicular to the axis of q. $[S\lambda\mu=0.]$

2. Q_8: *The i-j-k Group,* viz:

$$\pm 1,\ \pm i,\ \pm j,\ \pm k.$$

This is a special case of J_{4n}.

Q_{24}.

The Double-Tetrahedron Group.

$$i^\epsilon, \qquad j^\epsilon, \qquad k^\epsilon,$$

$$\left(\frac{1+i+j+k}{2}\right)^\eta,\quad \left(\frac{1-i-j+k}{2}\right)^\eta,\quad \left(\frac{1+i-j-k}{2}\right)^\eta,\quad \left(\frac{1-i+j-k}{2}\right)^\eta;$$

$$\epsilon=1,\ 2,\ 3,\ 4;\quad \eta=1,\ 2,\ 3,\ 4,\ 5,\ 6;\quad N=24.$$

Q_{48}.

The Double-Oktahedron Group.

$$\left(\frac{1+i}{\sqrt{2}}\right)^\epsilon,\quad \left(\frac{1+j}{\sqrt{2}}\right)^\epsilon,\quad \left(\frac{1+k}{\sqrt{2}}\right)^\epsilon;$$

$$\left(\frac{1+i+j+k}{2}\right)^\eta,\quad \left(\frac{1-i-j+k}{2}\right)^\eta,\quad \left(\frac{1+i-j-k}{2}\right)^\eta,\quad \left(\frac{1-i+j-k}{2}\right)^\eta;$$

$$\left(\frac{j+k}{\sqrt{2}}\right)^\zeta,\quad \left(\frac{k+i}{\sqrt{2}}\right)^\zeta,\quad \left(\frac{i+j}{\sqrt{2}}\right)^\zeta,$$

$$\left(\frac{j-k}{\sqrt{2}}\right)^\zeta,\quad \left(\frac{k-i}{\sqrt{2}}\right)^\zeta,\quad \left(\frac{i-j}{\sqrt{2}}\right)^\zeta;$$

$$\epsilon=1,\ \ldots\ 8;\quad \eta=1,\ \ldots\ 6;\quad \zeta=1,\ \ldots\ 4;\quad N=48.$$

Q_{120}.

1. *The Double-Ikosahedron Group.*

$$
\begin{array}{ccc}
i^{\epsilon} & j^{\epsilon} & k^{\epsilon} \\
\left(\frac{i+e'j+ek}{2}\right)^{\epsilon}, & \left(\frac{j+e'k+ei}{2}\right)^{\epsilon}, & \left(\frac{k+e'i+ej}{2}\right)^{\epsilon}, \\
\left(\frac{i-e'j+ek}{2}\right)^{\epsilon}, & \left(\frac{j-e'k-ei}{2}\right)^{\epsilon}, & \left(\frac{k+e'i-ej}{2}\right)^{\epsilon}, \\
\left(\frac{i+e'j-ek}{2}\right)^{\epsilon}, & \left(\frac{j-e'k+ei}{2}\right)^{\epsilon}, & \left(\frac{k-e'i-ej}{2}\right)^{\epsilon}, \\
\left(\frac{i-e'j-ek}{2}\right)^{\epsilon}, & \left(\frac{j+e'k-ei}{2}\right)^{\epsilon}, & \left(\frac{k-e'i+ej}{2}\right)^{\epsilon};
\end{array}
$$

$$
\left(\frac{1+i+j+k}{2}\right)^{\eta}, \quad \left(\frac{1-i-j+k}{2}\right)^{\eta}, \quad \left(\frac{1+i-j-k}{2}\right)^{\eta}, \quad \left(\frac{1-i+j-k}{2}\right)^{\eta},
$$

$$
\begin{array}{ccc}
\left(\frac{1+ej+e'k}{2}\right)^{\eta}, & \left(\frac{1+ek+e'i}{2}\right)^{\eta}, & \left(\frac{1+ei+e'j}{2}\right)^{\eta}, \\
\left(\frac{1+ej-e'k}{2}\right)^{\eta}, & \left(\frac{1+ek-e'i}{2}\right)^{\eta}, & \left(\frac{1+ei-e'j}{2}\right)^{\eta}; \\
\left(\frac{e+e'j+k}{2}\right)^{\zeta}, & \left(\frac{e+e'k+i}{2}\right)^{\zeta}, & \left(\frac{e+e'i+j}{2}\right)^{\zeta}, \\
\left(\frac{e+e'j-k}{2}\right)^{\zeta}, & \left(\frac{e+e'k-i}{2}\right)^{\zeta}, & \left(\frac{e+e'i-j}{2}\right)^{\zeta};
\end{array}
$$

$$
\epsilon=1, \ldots . 4; \quad \eta=1, \ldots . 6; \quad \zeta=1, \ldots . 10; \quad N=120.
$$

$$
e=\frac{1+\sqrt{5}}{2}, \quad e'=\frac{1-\sqrt{5}}{2}.
$$

2. Q'_{120}: A second Double-Ikosahedron Group differing from the above by the interchange of e with e'.

The groups Q_8 and Q_{24} are contained in Q_{48}, and Q_8 in Q_{24}, as permutable sub-groups.* Also a double-pyramid group of the $4m^{\text{th}}$ order contains a cyclotomic group of the $2m^{\text{th}}$ order as a *permutable* (ausgezeichnet) sub-group. The group Q_{120} contains no *permutable* sub-group except the group ± 1 of the 2^{d} order.

The vectors of the quaternions of these various groups represent geometrically the axes of symmetry of the corresponding regular polyhedra; that is, they are the diameters of the circumscribed sphere, which pass either through summits, or middle points of faces, or middle points of edges.

* C. Jordan, "Traité des substitutions," pp. 1–50; and W. Dyck, "Gruppe und Irrationalität regulärer Riemann'schen Flächen," *Math. Ann.* Bd. XVII, p. 481.

The Groups of Movements of the Regular Polyhedra in three-dimensional Space.

It is well known that any displacement of a sphere about its centre as a fixed point can be replaced by a simple rotation about a determinate axis, and that such a rotation is represented by the operator $q\,(\ )\,q^{-1}$, q being a quaternion whose axis is the axis of rotation and the double of whose angle is the amount of rotation. From the foregoing discussion, therefore, it follows immediately that all the finite groups of automorphic transformations of the sphere are represented by the various groups of operations of the form $q\,(\ )\,q^{-1}$; when, for a given group, q assumes all the values of the different quaternions in some one of the above enumerated quaternion groups. The linear transformations of Gordan are therefore represented in the following scheme.

$I_m\,(\ )\,I_m^{-1}$. If q assume all the values of the quaternions of a cyclotomic group of the m^{th} or $2m^{th}$ order, m being odd, then the operators of $q\,(\ )\,q^{-1}$, $q^2\,(\ )\,q^{-2}$, ... $q^m\,(\ )\,q^{-m}$ represent the m movements, or rotations about the axis of q, which transport any vertex of a polygon of m sides—whose plane is perpendicular to the axis of q—into all the other vertices of the polygon successively. These rotations I call the automorphic or self-congruent movements of the polygon.*

$J_{4n}\,(\ )\,J_{4n}^{-1}$. If q assume successively the values of the quaternions of a double-pyramid group of the order $4n$, the resulting rotations $q\,(\ )\,q^{-1}$ will represent all the self-congruent movements (*Bewegungen in sich*) of a double pyramid having in all $4n$ faces.

$Q_{24}\,(\ )\,Q_{24}^{-1}$. If q assume successively the values of the quaternions of the group Q_{24}, then the operators $q\,(\ )\,q^{-1}$ represent the 12 self-congruent movements of the tetrahedron, such that its four summits

$$\frac{i+j+k}{\sqrt{3}}, \qquad \frac{-i-j+k}{\sqrt{3}}, \qquad \frac{i-j-k}{\sqrt{3}}, \qquad \frac{-i+j-k}{\sqrt{3}},$$

its four centres of faces

$$-\frac{i+j+k}{\sqrt{3}}, \qquad -\frac{-i-j+k}{\sqrt{3}}, \qquad -\frac{i-j-k}{\sqrt{3}}, \qquad -\frac{-i+j-k}{\sqrt{3}},$$

and its six middle points of edges

$$\pm i, \qquad \pm j, \qquad \pm k,$$

are separately permuted with each other.

* I call a figure self-congruent, when without undergoing any change as a whole, like parts—as points with points, lines with lines, etc.—are interchanged. Self-congruent movements or rotations are those which produce such an interchange of like elements ; thus all the motions of a sphere about its centre as a fixed point are self-congruent movements.

$Q_{48}(\,)\,Q_{48}^{-1}$. If the quaternions of the group Q_{48} be substituted for q in $q(\,)\,q^{-1}$, the resulting transformations will be the 24 self-congruent movements of the oktahedron. The summits of this oktahedron are

$$\pm i, \qquad \pm j, \qquad \pm k;$$

its centres of faces are

$$\pm\frac{i+j+k}{\sqrt{3}}, \quad \pm\frac{-i-j+k}{\sqrt{3}}, \quad \pm\frac{i-j-k}{\sqrt{3}}, \quad \pm\frac{-i+j-k}{\sqrt{3}};$$

and its middle points of edges are

$$\pm\frac{j\pm k}{\sqrt{2}}, \quad \pm\frac{k\pm i}{\sqrt{2}}, \quad \pm\frac{i\pm j}{\sqrt{2}};$$

and the movements in question permute these *like* elements with each other.

$Q_{120}(\,)\,Q_{120}^{-1}$. Finally, if in $q(\,)\,q^{-1}$, q be made to assume all the values in Q_{120}, the result will be the 60 self-congruent movements of the ikosahedron. We have here for summits

$$\pm\frac{\epsilon' j\pm k}{\epsilon'\sqrt{5}}, \quad \pm\frac{\epsilon' k\pm i}{\epsilon'\sqrt{5}}, \quad \pm\frac{\epsilon' i\pm j}{\epsilon'\sqrt{5}};$$

for centres of faces

$$\pm\frac{i+j+k}{\sqrt{3}}, \quad \pm\frac{-i-j+k}{\sqrt{3}}, \quad \pm\frac{i-j-k}{\sqrt{3}}, \quad \pm\frac{-i+j-k}{\sqrt{3}},$$

$$\pm\frac{\epsilon j\pm\epsilon' k}{\sqrt{3}}, \quad \pm\frac{\epsilon k\pm\epsilon' i}{\sqrt{3}}, \quad \pm\frac{\epsilon i\pm\epsilon' j}{\sqrt{3}};$$

and for middle points of edges

$$\pm i, \qquad \pm j, \qquad \pm k,$$

$$\pm\frac{i\pm\epsilon' j\pm\epsilon k}{2}, \quad \pm\frac{j\pm\epsilon' k\pm\epsilon i}{2}, \quad \pm\frac{k\pm\epsilon' i\pm\epsilon j}{2}.$$

The above scheme contains all the possible finite groups of automorphic linear transformations of the sphere, or what is the same thing, of the plane of complex variables.

SCHWARZBACH, SAXONY, *September*, 1881.

The Counter-Pedal Surface of the Ellipsoid.

By Thomas Craig, *Johns Hopkins University.*

The particular surface investigated in the present paper is merely one of a class whose properties do not seem to have been studied; there is at least no reference to the surfaces in any works that I am acquainted with; further, Prof. Cayley expresses his belief that the subject is quite new. The definition of a counter-pedal surface is obtained from the definition of a pedal surface by a very slight change of the wording, thus: a pedal surface is the locus of the intersection of the tangent planes to a given surface with straight lines drawn from a given point parallel to the normal; the counter-pedal surface is the locus of the intersection of the normals to a given surface with planes drawn through a given point parallel to the tangent planes. We can have in like manner pedal curves and counter-pedal curves. Problems connected with counter-pedals, whether curves or surfaces, will in general be more difficult than those connected with pedals, as it would seem that in general the degree of the counter-pedal is higher than that of the pedal. Before taking up the analytical part of the work I must first express my great obligations to Prof. Cayley for the assistance he has given me in this connection, especially in finding the equation of the surface. I had found the equation of the surface, but in a complicated form and encumbered with a factor of the sixth degree, which I knew must exist but could not find. On presenting my difficulty to Prof. Cayley, he was good enough to work out for me the equation of the surface in another and simpler form. The investigation below to find the equation of the surface is, with some insignificant changes, entirely due to Prof. Cayley. I have also received assistance from him on many other minor points in working out the properties of this surface. I have not as yet done much on the general theory of counter-pedal surfaces and curves, but if one may judge anything from analogy in the case of the counter-pedals of the ellipse and ellipsoid, these new curves and surfaces seem to be more

closely related to negative pedals than to the direct pedals. For example, the degree of the negative and counter-pedals of the ellipse is six; the degree of the negative and counter-pedals of the ellipsoid is ten; while the degree of the pedals is in each case four. In both of these cases the pole is supposed to be at the centre. If for the ellipse the pole is taken at the focus, the negative pedal is a curve of the fourth degree having the lines $x^2+y^2=0$ for stationary tangents.

The counter-pedal surface of the ellipsoid, as considered in the following, is the locus of the intersections of central planes parallel to the tangent planes of the ellipsoid with the normals at the corresponding points of contact. The ellipsoid is given by the equation

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1.$$

The length of the central perpendicular upon the tangent plane to the ellipsoid at any point may be denoted by P; then

$$\frac{1}{P^2}=\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}+\frac{\zeta^2}{c^4};$$

further write

$$\alpha,\ \beta,\ \gamma=b^2-c^2,\ c^2-a^2,\ a^2-b^2.$$

Denote by x, y, z the coördinates of a point on the counter-pedal corresponding to a point ξ, η, ζ on the ellipsoid, then it is obvious that these are given by the equations

$$x=\xi\left(1-\frac{P^2}{a^2}\right),\quad y=\eta\left(1-\frac{P^2}{b^2}\right),\quad z=\zeta\left(1-\frac{P^2}{c^2}\right).$$

In order to find the equation of the counter-pedal we have to eliminate ξ, η, ζ between these equations and

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1.$$

It is to be noticed that these equations give

$$x\xi+y\eta+z\zeta=x^2+y^2+z^2,$$

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}=0,$$

$$\frac{\xi}{a^2}(y\zeta-z\eta)+\frac{\eta}{b^2}(z\xi-x\zeta)+\frac{\zeta}{c^2}(x\eta-y\xi)=0.$$

The geometrical meaning of these equations is obvious; one should be able to eliminate between these and the equation of the ellipsoid and find the equa-

tion of the surface. I was, however, not able to do it. The result was of the degree sixteen, instead of ten, the proper degree; but I could not find the required factor. Instead of $-P^2$ it will be convenient to write merely t, then

$$x=\xi\left(\frac{a^2+t}{a^2}\right), \quad y=\eta\left(\frac{b^2+t}{b^2}\right), \quad z=\zeta\left(\frac{c^2+t}{c^2}\right);$$

from these by aid of

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1$$

we derive

$$\frac{a^2x^2}{(a^2+t)^2}+\frac{b^2y^2}{(b^2+t)^2}+\frac{c^2z^2}{(c^2+t)^2}=1,$$

and

$$\frac{x^2}{(a^2+t)^2}+\frac{y^2}{(b^2+t)^2}+\frac{z^2}{(c^2+t)^2}=-\frac{1}{t}.$$

For greater convenience replace x^2, y^2, z^2, a^2, b^2, c^2 by x, y, z, a, b, c, then to find the required equation we have to eliminate t between

$$\frac{ax}{(a+t)^2}+\frac{by}{(b+t)^2}+\frac{cz}{(c+t)^2}=1$$

$$\frac{x}{(a+t)^2}+\frac{y}{(b+t)^2}+\frac{z}{(c+t)^2}=-\frac{1}{t}.$$

Multiply the second of these by t and add to the first; this gives

$$\frac{x}{a+t}+\frac{y}{b+t}+\frac{z}{c+t}=0,$$

which is to be combined with

$$\frac{ax}{(a+t)^2}+\frac{by}{(b+t)^2}+\frac{cz}{(c+t)^2}=1.$$

We have thus to eliminate t between a sextic and a quadratic equation. The method followed by Prof. Cayley to effect this elimination is as follows: Expanding the quadratic it is

$$(x+y+z)t^2+[(b+c)x+(c+a)y+(a+b)z]t+bcx+cay+abz=0.$$

Write this as

$$Pt^2+Qt+R=0.$$

Denote by t_1 and t_2 the roots of this quadratic, then we have

$$Pt^2+Qt+R=P(t-t_1)(t-t_2),$$

$$t_1+t_2=-\frac{Q}{P}, \quad t_1t_2=\frac{R}{P}.$$

The minimum value of t is of course $-a$, and its maximum value $-c$. Make $t = -a$, then we have

$$a^2(x+y+z) - a(\overline{b+c}\,x + \overline{c+a}\,y + \overline{a+b}\,z) + bcx + cay + abz = P(a+t_1)(a+t_2)$$

and obviously

$$P(a+t_1)(a+t_2) = -\beta\gamma x$$
$$P(b+t_1)(b+t_2) = -\gamma\alpha y$$
$$P(c+t_1)(c+t_2) = -\alpha\beta z$$

The quantity $(t_1 - t_2)^2$ will be needed; this is given by

$$(t_1 - t_2)^2 = \frac{Q^2 - 4PR}{P^2}.$$

Substituting for P, Q, R their values, this is

$$(t_1 - t_2)^2 = \frac{\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2 - 2\beta\gamma yz - 2\alpha\beta xy - 2\gamma\alpha zx}{(x+y+z)^2}.$$

or say

$$(t_1 - t_2)^2 = \frac{V}{P^2}.$$

The values of t_1 and t_2, substituted in the sextic equation, will lead to the expression required for the equation of the counter-pedal. The sextic expanded is

$$(b+t)^2(c+t)^2ax + (c+t)^2(a+t)^2by + (b+t)^2(a+t)^2cz - (a+t)^2(b+t)^2(c+t)^2 = 0.$$

Write this as

$$Aax + Bby + Ccz - D = 0;$$

denote by the suffix 1 or 2 the result of substituting for t its values t_1 and t_2; multiplying together the two identical equations thus obtained we have

$$(A_1ax + B_1by + C_1cz - D_1)(A_2ax + B_2by + C_2cz - D_2) = 0;$$

this expanded is

$$A_1A_2a^2x^2 + B_1B_2b^2y^2 + C_1C_2c^2z^2 + D_1D_2$$
$$+ (B_1C_2 + B_2C_1)bcyz + (A_1C_2 + A_2C_1)cazx + (A_1B_2 + A_2B_1)abxy$$
$$- (A_1D_2 + A_2D_1)ax - (B_1D_2 + B_2D_1)by - (C_1D_2 + C_2D_1)cz = 0.$$

Using the above equations giving the values of x, y, z, the coefficients in this are readily found.

$$A_1A_2 = (b+t_1)^2(b+t_2)^2(c+t_1)^2(c+t_2)^2 = \frac{\alpha^2\beta^2\gamma^2}{P^4}\cdot\alpha^2y^2z^2$$

$$D_1D_2 = (a+t_1)^2(a+t_2)^2 \ldots\ldots (c+t_2)^2 = \frac{\alpha^2\beta^2\gamma^2}{P^6}\cdot\alpha^2\beta^2\gamma^2x^2y^2z^2$$

$$
\begin{aligned}
B_1C_2+B_2C_1 &= (c+t_1)^2(a+t_1)^2(a+t_2)^2(b+t_2)^2+(c+t_2)^2(a+t_2)^2(a+t_1)^2(b+t_1)^2 \\
&= (a+t_1)^2(a+t_2)^2\left[(c+t_1)^2(b+t_2)^2+(c+t_2)^2(b+t_1)^2\right] \\
&= (a+t_1)^2(a+t_2)^2\left[a^2(t_1-t_2)^2+2\,\overline{b+t_1}\,.\,\overline{b+t_2}\,.\,\overline{c+t_1}\,.\,\overline{c+t_2}\right] \\
&= \frac{\beta^2\gamma^2x^2}{P^2}\left[\frac{a^2\nabla}{P^2}+2\frac{a^2\beta\gamma yz}{P^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
A_1D_2+A_2D_1 &= (b+t_1)^2(c+t_1)^2(a+t_2)^2(b+t_2)^2(c+t_2)^2 \\
&\qquad + (b+t_2)^2(c+t_2)^2(a+t_1)^2(b+t_1)^2(c+t_1)^2 \\
&= (b+t_1)^2(b+t_2)^2(c+t_1)^2(c+t_2)^2\left[(a+t_2)^2+(a+t_1)^2\right] \\
&= \frac{\alpha^2\beta^2\gamma^2}{P^4}\cdot a^2y^2z^2\left[(t_1-t_2)^2+2(a+t_1)(a+t_2)\right] \\
&= \frac{\alpha^2\beta^2\gamma^2}{P^4}\cdot a^2y^2z^2\left[\frac{\nabla}{P^2}-\frac{2\beta\gamma x}{P}\right]
\end{aligned}
$$

The remaining values are written down by symmetry. Substitution of the values in the above equation gives

$$
\frac{\alpha^2\beta^2\gamma^2}{P^4}x^2y^2z^2\left[a^2\alpha^2+b^2\beta^2+c^2\gamma^2\right]+\frac{\alpha^2\beta^2\gamma^2}{P^6}x^2y^2z^2
$$

$$
+\,xyz\frac{\alpha^2\beta^2\gamma^2}{P^4}\left\{(bcx+cay+abz)\nabla+2xyz\left[bc\beta\gamma+ca\gamma\alpha+ab\alpha\beta\right]\right\}
$$

$$
-\,xyz\frac{\alpha^2\beta^2\gamma^2}{P^6}\left\{\nabla(a\alpha^2yz+b\beta^2zx+c\gamma^2xy)-2xyz\,.\,\alpha\beta\gamma P\left[a\alpha+b\beta+c\gamma\right]\right\}=0\,.
$$

The whole expression divides by the factor

$$
xyz\,.\,\frac{\alpha^2\beta^2\gamma^2}{P^4}\,;
$$

effect this division, then multiply out by P^2 and reduce further by noting th $a\alpha+b\beta+c\gamma=0$. The terms in the result containing P^2xyz as a factor a easily seen to be

$$
P^2xyz\left[a^2\alpha^2+b^2\beta^2+c^2\gamma^2+2\beta\gamma bc+2\gamma\alpha ca+2\alpha\beta ab\right]=P^2xyz(a\alpha+b\beta+c\gamma)^2=
$$

The terms remaining are now

$$
\alpha^2\beta^2\gamma^2xyz-(a\alpha^2yz+b\beta^2zx+c\gamma^2xy)\nabla+P^2(bcx+cay+abz)\nabla=0\,.
$$

This is the equation of a skew surface of the fifth degree. The princip section in the plane of xy is the cubic

$$
(a-b)^2xy-(bx+ay)(x+y)^2=0\,.
$$

This has an asymptote

$$
bx+ay+ab=0
$$

which meets the curve in

$$
ax+by=0\,.
$$

There is further an asymptotic parabola

$$(x+y)^2 = ax + by + ab,$$

and this meets the curve in

$$x + y = -\frac{ab}{a+b}$$

$$bx + ay = \left(\frac{ab}{a+b}\right)^2.$$

The discussion of this surface is reserved for the present, but we may just notice the values of the coördinates of a point on any of the principal sections corresponding to a given value of t. We have

$$\frac{ax}{(a+t)^2} + \frac{by}{(b+t)^2} + \frac{cz}{(c+t)^2} = 1,$$

$$\frac{x}{a+t} + \frac{y}{b+t} + \frac{c}{c+t} = 0.$$

For the section in the plane $x = 0$ we may write

$$y = M(b+t), \quad z = -M(c+t),$$

then

$$M\left(\frac{b}{b+t} - \frac{c}{c+t}\right) = 1,$$

and so

$$M = \frac{(b+t)(c+t)}{(b-c)t}.$$

Introducing this value of M we have the values of y, and z corresponding to $x = 0$. Similarly the coördinates of a point in the other two planes may be given. Arranged in tabular form these are

$x = 0$	$x = -\frac{(c+t)(a+t)^2}{(c-a)t}$	$x = \frac{(a+t)^2(b+t)}{(a-b)t}$
$y = \frac{(b+t)^2(c+t)}{(b-c)t}$	$y = 0$	$y = -\frac{(a+t)(b+t)^2}{(a-b)t}$
$z = -\frac{(b+t)(c+t)^2}{(b-c)t}$	$z = \frac{(c+t)^2(a+t)}{(c-a)t}$	$z = 0$

The equations of the sections of the skew surface corresponding to these sets of values of x, y, z are

$$(b-c)^2yz - (cy+bz)(y+z)^2 = 0,$$
$$(c-a)^2zx - (az+cx)(z+x)^2 = 0,$$
$$(a-b)^2xy - (bx+ay)(x+y)^2 = 0.$$

If in the equation of the skew surface we replace x, y, z, a, b, c by the squares of these quantities we have at once the required equation of the counter-pedal. The correspondence between these two surfaces is such that to every

generator on the ruled surface corresponds a certain curve on the counter-ped The curve and generator are found by giving t a certain constant value in bc cases. The lines on the ellipsoid corresponding to a given value of t are read found. Required to find the line on

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1$$

for which t, *i. e.* $-P^2$, has a constant value, say $-t_0$: this gives

$$\frac{\xi^2}{a^4} + \frac{\eta^2}{b^4} + \frac{\zeta^2}{c^4} = \frac{1}{t_0};$$

the required curve is then the intersection of the two ellipsoids

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1$$

$$\frac{t_0\xi^2}{a^4} + \frac{t_0\eta^2}{b^4} + \frac{t_0\zeta^2}{c^4} = 1.$$

The projections of this curve on the coördinate planes are evidently coni the projection on the plane of xz for example is the hyperbola

$$\frac{\xi^2}{a^2} \cdot \frac{a^2 - b^2}{a^2} - \frac{\zeta^2}{c^2} \cdot \frac{b^2 - c^2}{c^2} = \frac{t_0 - b^2}{t_0}.$$

From their definition, these curves are obviously Poinsot's pollhodes. For t value $t_0 = b^2$ the curve on the ellipsoid is the *separating* pollhode; its projecti on the plane zx is the pair of lines

$$\frac{\xi^2}{a^2} \cdot \frac{a^2 - b^2}{a^2} - \frac{\zeta^2}{c^2} \cdot \frac{b^2 - c^2}{c^2} = 0.$$

The pollhode itself consists of two ellipses whose planes pass through the a of y. The curves on the counter-pedal corresponding to a certain constant val of t, *i. e.* $-P^2$, are found by laying off on the normals to the ellipsoid along t corresponding pollhode the length P. The extremities of these lines also lie that parallel to the ellipsoid for which the modulus $= P$; the lines on the count pedal are then the intersections of the inner sheet of the parallel surface w the counter-pedal. These are only partial intersections, and I have not been al to determine the degree of the curve.

The equation of the counter-pedal is obtained from that of the skew surf as above indicated, and is

$$\begin{aligned}
&\alpha^2\beta^2\gamma^2x^2y^2z^2 \\
&\quad - (a^2\alpha^2y^2z^2 + b^2\beta^2z^2x^2 + c^2\gamma^2x^2y^2)(\alpha^2x^4 + \beta^2y^4 + \gamma^2z^4 - 2\alpha\beta xy - 2\beta\gamma yz - 2\gamma\alpha zx \\
&\quad + (x^2 + y^2 + z^2)^2 (b^2c^2x^2 + c^2a^2y^2 + a^2b^2z^2)(\alpha^2x^4 + \beta^2y^4 + \gamma^2z^4 \\
&\qquad\qquad - 2\beta\gamma yz - 2\gamma\alpha zx - 2\alpha\beta xy) =
\end{aligned}$$

The counter-pedal is therefore of the tenth degree—the same degree as that of the negative pedal of the ellipsoid. The terms are grouped into those of the sixth, eighth, and tenth degrees respectively. Writing

$$A=[\Sigma x^2]^2\,,\ B=\Sigma a^2x^4\,,\ C=\Sigma b^2c^2x^2\,,$$
$$D=\Sigma a^2\alpha^2y^2z^2\,,\ E=2\Sigma\alpha\beta x^2y^2\,,\ G=\alpha^2\beta^2\gamma^2x^2y^2z^2\,,$$

the equation may be written

$$G+(B-E)(AC-D)=0\,,$$

or, writing

$$B-E=-L\,,\qquad AC-D=M\,,$$

the equation is

$$G-LM=0\,.$$

The L is the ∇ of the above. For $G=0$ with $M=0$ we have the principal sections of the surface—complete in the planes $z=0$ and $x=0$, but as we shall see not complete in $y=0$. These principal sections are the counter-pedals of the principal sections of the ellipsoid together with, in the plane $y=0$, a certain pair of straight lines. The equations of the counter-pedals of the principal ellipses are

$$(b^2-c^2)^2y^2z^2-(c^2y^2+b^2z^2)(y^2+z^2)^2=0\,,$$
$$(c^2-a^2)^2z^2x^2-(a^2z^2+c^2x^2)(z^2+x^2)^2=0\,,$$
$$(a^2-b^2)^2x^2y^2-(b^2x^2+a^2y^2)(x^2+y^2)^2=0\,.$$

The coördinates of a point on each of these sextic curves are given by the table

$x^2=0$	$x^2=-\dfrac{(c^2+t)(a^2+t)^2}{(c^2-a^2)t}$	$x^2=\dfrac{(a^2+t)^2(b^2+t)}{(a^2-b^2)t}$
$y^2=\dfrac{(b^2+t)^2(c^2+t)}{(b^2-c^2)t}$	$y^2=0$	$y=-\dfrac{(a^2+t)(b^2+t)^2}{(a^2-b^2)t}$
$z^2=-\dfrac{(b^2+t)(c^2+t)^2}{(b^2-c^2)t}$	$z^2=\dfrac{(c^2+t)^2(a^2+t)}{(c^2-a^2)t}$	$z=0$

It is important to notice that these values go in pairs. For values of $-t<b^2$ the first and second columns give real values of x, y and z, while the third gives imaginary values; for $-t=b^2$ the first and third columns vanish; for values of $-t>b^2$ the first column is imaginary while the second and third are real. This remark is important in making a drawing or model of the surface, as it shows just what points in the three cross sections are to be united.

If in the equation of the surface we make

$$G = 0 \text{ and } L = 0$$

we have for the three possible cases a pair of straight lines in each of the plane These are

$$(\alpha x^2 - \beta y^2)^2 = 0 \quad \text{in } z = 0\,;$$
$$(\beta y^2 - \gamma z^2)^2 = 0 \quad \text{in } x = 0\,,$$
$$(\gamma z^2 - \alpha x^2)^2 = 0 \quad \text{in } y = 0\,.$$

Of these only the last,

$$\alpha x^2 - \gamma z^2 = 0\,,$$

is real and completes the principal section in the plane $y = 0$. The prii cipal sections of the surface shown in isometric projections are given in figure

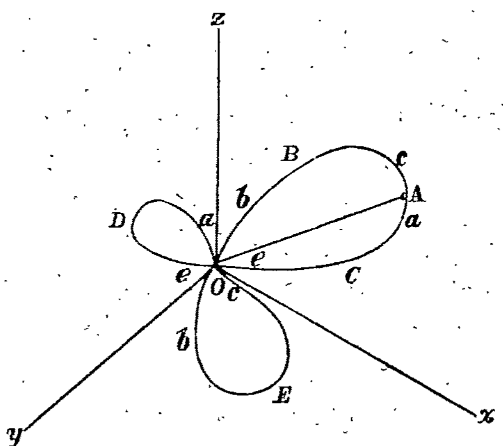


Fig.1.

The portion OBA of the loop in the plane $y = 0$ corresponds to values ‹ t between $-b^2$ and $-c^2$ and is joined to the corresponding points in the loop *I* the values of the corresponding points in OBA and E are found from the abo table. The portion OCA is joined in like manner to D. The small letters ind cate corresponding points and serve to show completely how a drawing or mod of the surface may be constructed.

In making a drawing of the surface it will of course be desirable to woi out the numerical values of a few of the corresponding points though the lin joining them will have to be filled in by the eye—unless indeed one chooses go through a very tedious computation to find the intermediate points throug which to trace them.

The lines

$$\alpha x^2 - \gamma z^2 = 0$$

are clearly double lines on the surface; a section of the surface through the point a of Fig. 1 will consist of two curves like Fig. 2.

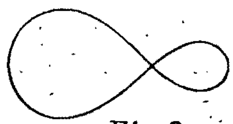


Fig. 2.

This section gives only eight real points of intersection with a right line. There are four real singular points of the nature of triple points in the plane of $y = 0$ and an equal number of imaginary ones in the other two planes; the origin is a singular point of a very high order and will hardly repay study. The double lines

$$\alpha x^2 - \gamma z^2 = 0$$

on the counter-pedal correspond to the separating pollhode on the ellipsoid; the projection of these lines has been shown to be

$$\frac{x^2}{a^2} \cdot \frac{\gamma}{a^2} - \frac{z^2}{c^2} \cdot \frac{\alpha}{c^2} = 0;$$

the angle between these projections and the double lines being called θ, we have

$$\cos\theta = (a^2 + c^2)\sqrt{\alpha\gamma} \div \sqrt{\alpha\gamma(a^4 + c^4) + a^4\alpha^2 + c^4\gamma^2}.$$

This is a right angle only for ellipsoids of revolution, as we can only have $\cos\theta = 0$ for $\alpha = 0$ or $\gamma = 0$. The projection and the double line will coincide with each other for $\cos\theta = 1$; this gives

$$b^2 = c^2\,\frac{a^2 - c^2}{a^2 + c^2}.$$

The double lines are at right angles for an ellipsoid whose axes are connected by the relation

$$b^2 = \frac{2a^2c^2}{a^2 + c^2}.$$

I thought that the umbilics of the ellipsoid might correspond to the above mentioned triple points or the counter-pedal; on testing I found that they did not correspond, though there is a singular relation between the coördinates of the points corresponding to the umbilics and the coördinates of the triple points.

The coördinates of the umbilics on the ellipsoid are

$$\xi^2 = -a^2 \frac{\gamma}{\beta}, \quad \eta = 0, \quad \zeta^2 = -c^2 \frac{\alpha}{\beta};$$

the value of t for these points is $-\frac{a^2c^2}{b^2}$; substituting this in the second column of the above table we have

$$x^2 = -\frac{\gamma a^2}{b^2\beta}, \quad y = 0, \quad z^2 = -\frac{\gamma^2 a}{b^2\beta}.$$

The coördinates of the triple points on the counter-pedal are found by making $t = -b^2$; they are

$$x^2 = -\frac{\gamma^2 a}{b^2\beta}, \quad y = 0, \quad z^2 = -\frac{\gamma a^2}{b^2\beta}.$$

The values of x and z for the two series of points are just interchanged. The condition that the umbilics on the ellipsoid should correspond to the triple points on the counter-pedal is

$$b^2 = \frac{a^2 + c^2}{2}.$$

Besides the two straight double lines on the surface there are two other double lines whose projections are of a quadrifoil shape, the curves in space having the same shape only bent to fit the surface.

The existence of these lines is readily inferred from Fig. 1; they are produced by the intersection of the portion of the surface joining the loop E to the partial loop OBA with the portion which joins the loop D to the partial loop OCA. The degree of these nodal lines I have not been able to determine.

The lines of curvature on the counter-pedal to the ellipsoid are its intersections with the counter-pedals to the two confocal hyperholoids. I have not been able to deduce any results of consequence concerning the curvature of this surface, but in the next few pages I have given the values of all of the quantities employed by Gauss in his investigations on the curvature of surfaces. These values will certainly be necessary in any future study of the curvature of the counter-pedal, so it is worth while giving them here, although I have not been able to determine in a simple manner the radii of curvature.

Denoting by u and v the curvilinear coördinates of a point on the surface of the ellipsoid, we have

$$\begin{aligned} -\beta\gamma\xi^2 &= a^2(a^2+u)(a^2+v), \\ -\gamma\alpha\eta^2 &= b^2(b^2+u)(b^2+v), \\ -\alpha\beta\zeta^2 &= c^2(c^2+u)(c^2+v). \end{aligned}$$

The element of area of the counter-pedal is

$$d\Sigma = \sqrt{\mathfrak{E}\mathfrak{G} - \mathfrak{F}^2}\,du\,dv, \; = U\,du\,dv.$$

Now

$$U^2 = \frac{P^4}{16P_1^2P_2^2}\left\{\left(\frac{1}{P}+\frac{P}{u}\right)^2\left(\frac{1}{P}+\frac{P}{v}\right)^2 + \frac{P_2^2}{v^2}\left(\frac{1}{P}+\frac{P}{u}\right)^2 + \frac{P_1^2}{u^2}\left(\frac{1}{P}+\frac{P}{v}\right)^2\right\}.$$

This is easily brought into the form

$$U^2 = \frac{1}{16}\,\frac{(P^2+u)^2(P^2+v)^2}{P_1^2P_2^2u^2v^2}\left[1 + \frac{P^2P_1^2}{(P^2+u)^2} + \frac{P^2P_2^2}{(P^2+v)^2}\right],$$

or again

$$U^2 = \frac{1}{16}\,\frac{1}{P_1^2P_2^2}\left[\frac{(P^2+u)^2(P^2+v)^2}{u^2v^2} + \frac{P^2P_1^2\,(P^2+v)^2}{u^2v^2} + \frac{P^2P_2^2(P^2+u)^2}{u^2v^2}\right],$$

or say

$$U^2 = \frac{1}{16}\,\frac{T^2}{P_1^2P_2^2};$$

the element of area is now

$$d\Sigma = \frac{1}{4}\,\frac{T}{P_1P_2}\,du\,dv,$$

and for the total area we must have

$$\Sigma = 2\int_{-a^2}^{-b^2}\int_{-b^2}^{-c^2}\frac{T}{P_1P_2}\,du\,dv:$$

of course the area of the ellipsoid is

$$S = 2\int_{-a^2}^{-b^2}\int_{-b^2}^{-c^2}\frac{du\,dv}{P_1P_2}.$$

The ratio between corresponding elements of area on the ellipsoid and its counter-pedal is

$$\frac{d\Sigma}{dS} = T.$$

Denoting by $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ the direction-cosines of the normals to the ellipsoid and counter-pedal respectively at corresponding points, we have

$$T = \begin{vmatrix} \alpha_2, & \dfrac{dx}{d\xi}, & \dfrac{dx}{d\eta}, & \dfrac{dx}{d\zeta} \\ \beta_2, & \dfrac{dy}{d\xi}, & \dfrac{dy}{d\eta}, & \dfrac{dy}{d\zeta} \\ \gamma_2, & \dfrac{dz}{d\xi}, & \dfrac{dz}{d\eta}, & \dfrac{dz}{d\zeta} \\ 0 & \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}.$$

The verification of this value of T by working out the determinant would be a very difficult matter, as the values of α_3, β_3, γ_3, whether obtained directly from the equation of the surface or in the manner indicated below, are extremely complicated.

Using the notation employed by Salmon, *Geom. of three Dimensions*, write

$$A,\ B,\ C = \frac{d(\eta,\zeta)}{d(u,v)},\ \frac{d(\zeta,\xi)}{d(u,v)},\ \frac{d(\xi,\eta)}{d(u,v)}$$

referring to the ellipsoid; also

$$\mathfrak{F},\ \mathfrak{G},\ \mathfrak{D} = \frac{d(y,z)}{d(u,v)},\ \frac{d(z,x)}{d(u,v)},\ \frac{d(x,y)}{d(u,v)}$$

referring to the counter-pedal. For convenience I will write the coördinates x, y, z as

$$x = \xi\left(1+\frac{k}{a^2}\right) \&\text{c., } i.\ e.\ k = -P^2;$$

also write

$$\bar{a},\ \bar{b},\ \bar{c} = \left(1+\frac{k}{a^2}\right),\ \left(1+\frac{k}{b^2}\right),\ \left(1+\frac{k}{c^2}\right).$$

We find now at once

$$\mathfrak{F} = \bar{b}\bar{c}\frac{d(\eta,\zeta)}{d(u,v)} + \frac{\bar{c}\eta}{b^2}\frac{d(k,\zeta)}{d(u,v)} + \frac{\bar{b}\zeta}{c^2}\frac{d(\eta,k)}{d(u,v)}.$$

or

$$\mathfrak{F} = -\begin{vmatrix} -\bar{b}\bar{c}, & \dfrac{\bar{c}\eta}{b^2}, & \dfrac{\bar{b}\zeta}{c^2} \\ \dfrac{dk}{du}, & \dfrac{d\eta}{du}, & \dfrac{d\zeta}{du} \\ \dfrac{dk}{dv}, & \dfrac{d\eta}{dv}, & \dfrac{d\zeta}{dv} \end{vmatrix}$$

$$= A - \begin{vmatrix} -\left(\dfrac{k}{b^2}+\dfrac{k}{c^2}+\dfrac{k^2}{b^2c^2}\right), & \dfrac{\eta}{b^2}\left(1+\dfrac{k}{c^2}\right), & \dfrac{\zeta}{c^2}\left(1+\dfrac{k}{b^2}\right) \\ \dfrac{dk}{du}, & \dfrac{d\eta}{du}, & \dfrac{d\zeta}{du} \\ \dfrac{dk}{dv}, & \dfrac{d\eta}{dv}, & \dfrac{d\zeta}{dv} \end{vmatrix}$$

or finally

$$\mathfrak{F} = \mathfrak{A} - \begin{vmatrix} \frac{-k}{b^2c^2}\left(b^2+c^2\right), & \frac{\eta}{b^2}, & \frac{\zeta}{c^2} \\ \frac{dk}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix} - \frac{k}{b^2c^2} \begin{vmatrix} -k, & \eta, & \zeta \\ \frac{dk}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix}.$$

The values of $\mathfrak{G}$ and $\mathfrak{H}$ may of course be written down at once by symmetry.

There are three other quantities necessary in obtaining the expressions for the radii of curvature, &c., by Gauss's method, viz.

$$\mathfrak{E}' = \begin{vmatrix} \frac{d^2x}{du^2}, & \frac{d^2y}{du^2}, & \frac{d^2z}{du^2} \\ \frac{dx}{du}, & \frac{dy}{du}, & \frac{dz}{du} \\ \frac{dx}{dv}, & \frac{dy}{dv}, & \frac{dz}{dv} \end{vmatrix} \text{ \&c.,}$$

or

$$\mathfrak{E}' = \frac{d^2x}{du^2}\mathfrak{F} + \frac{d^2y}{du^2}\mathfrak{G} + \frac{d^2z}{du^2}\mathfrak{H}$$

$$\mathfrak{F}' = \frac{d^2x}{dudv}\mathfrak{F} + \frac{d^2y}{dudv}\mathfrak{G} + \frac{d^2z}{dudv}\mathfrak{H}$$

$$\mathfrak{G}' = \frac{d^2x}{dv^2}\mathfrak{F} + \frac{d^2y}{dv^2}\mathfrak{G} + \frac{d^2z}{dv^2}\mathfrak{H}.$$

The values of these expressed as determinants of the fourth order are easily found; they are

$$\mathfrak{E}' = \begin{vmatrix} \overline{abc}, & \frac{-\overline{bc}}{a^2}\xi, & \frac{-\overline{ca}}{b^2}\eta, & \frac{-\overline{ab}}{c^2}\zeta \\ \frac{d^2k}{du^2}, & \frac{d^2\xi}{du^2}, & \frac{d^2\eta}{du^2}, & \frac{d^2\zeta}{du^2} \\ \frac{dk}{du}, & \frac{d\xi}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\xi}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix} + 2\frac{dk}{du}\left\{\frac{1}{a^2}\frac{d\xi}{du}\mathfrak{F} + \frac{1}{b^2}\frac{d\eta}{du}\mathfrak{G} + \frac{1}{c^2}\frac{d\zeta}{du}\mathfrak{H}\right\}.$$

Writing $\overline{abc} = T + 1$, this becomes

$$\mathfrak{E}' = E' + 2\frac{dk}{du}\Sigma\frac{1}{a^2}\frac{d\xi}{du}V - \begin{vmatrix} -T, & \frac{\overline{bc}}{a^2}\xi, & \frac{\overline{ca}}{b^2}\eta, & \frac{\overline{ab}}{c^2}\zeta \\ \frac{d^2k}{du^2}, & \frac{d^2\xi}{du^2}, & \frac{d^2\eta}{du^2}, & \frac{d^2\zeta}{du^2} \\ \frac{dk}{du}, & \frac{d\xi}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\xi}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix}$$

The expression for $\mathfrak{G}'$ may be written down by symmetry; that for $\mathfrak{F}'$ is

$$\mathfrak{F}' = \frac{dk}{dv}\Sigma\frac{1}{a^2}\frac{d\xi}{du}V + \frac{dk}{du}\Sigma\frac{1}{a^2}\frac{d\xi}{dv}V - \begin{vmatrix} -T, & \frac{\overline{bc}}{a^2}\xi, & \frac{\overline{ca}}{b^2}\eta, & \frac{\overline{ab}}{c^2}\zeta \\ \frac{d^2k}{dudv}, & \frac{d^2\xi}{dudv}, & \frac{d^2\eta}{dudv}, & \frac{d^2\zeta}{dudv} \\ \frac{dk}{du}, & \frac{d\xi}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\xi}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix}$$

I have not been able to simplify these expressions any considerable amount.

If we denote by $\bar{x}, \bar{y}, \bar{z}$ the coördinates of a point on the *pedal* of the ellipsoid, it is easy to see that

$$\xi = x + \bar{x}, \quad \eta = y + \bar{y}, \quad \zeta = z + \bar{z}$$

or

$$x = \xi - \bar{x}, \quad y = \eta - \bar{y}, \quad z = \zeta - \bar{z}.$$

If $\bar{E}, \bar{F}, \bar{G}$ refer to the pedal, we have for the element of length on that surface

$$d\bar{\sigma}^2 = \bar{E}du^2 + 2\bar{F}dudv + \bar{G}dv^2.$$

It is quite easy to show that

$$\mathfrak{E} = E + \bar{E} + \frac{P^2}{2uP_1^2},$$

$$\mathfrak{G} = G + \bar{G} + \frac{P^2}{2vP_2^2},$$

$$\mathfrak{F} = \bar{F}.$$

and then for the element of length on the counter-pedal

$$d\sigma^2 = ds^2 + d\bar{\sigma}^2 + \frac{P^2}{2}\left(\frac{du^2}{uP_1^2} + \frac{dv^2}{vP_2^2}\right);$$

writing

$$\Lambda^2 = a^2b^2c^2,$$
$$\Lambda_1^2 = a^2 + u \,.\, b^2 + u \,.\, c^2 + u,$$
$$\Lambda_2^2 = a^2 + v \,.\, b^2 + v \,.\, c^2 + v,$$

the last term in $d\sigma^2$ becomes

$$\frac{P^2}{2}(u - v)\left[\frac{du^2}{\Lambda_1^2} - \frac{dv^2}{\Lambda_2^2}\right],$$

$$= \frac{\Lambda^2}{2}\left(\frac{1}{v} - \frac{1}{u}\right)\left[\frac{du^2}{\Lambda_1^2} - \frac{dv^2}{\Lambda_2^2}\right];$$

and so

$$d\sigma^2 = ds^2 + d\bar{\sigma}^2 + \frac{\Lambda^2}{2}\left(\frac{1}{v} - \frac{1}{u}\right)\left[\frac{du^2}{\Lambda_1^2} - \frac{dv^2}{\Lambda_2^2}\right].$$

It would be interesting to find an interpretation of this last term, but I see no means of doing it.

Denoting by $d\bar{\Sigma}$ the element of area on the pedal, we have

$$d\bar{\Sigma} = \sqrt{\bar{E}\bar{G} - \bar{F}^2}\,du\,dv;$$

also

$$\bar{E} = \frac{P^2}{4u^2} \cdot \frac{P^2 + P_1^2}{P_1^2},$$

$$\bar{G} = \frac{P^2}{4v^2} \cdot \frac{P^2 + P_2^2}{P_2^2},$$

$$\bar{F} = \frac{1}{4} \cdot \frac{P^2}{uv};$$

calling the quantity under the radical sign $\bar{V}^2$ we have

$$\bar{V}^2 = \frac{P^6}{16u^2v^2P_1^2P_2^2}\left[P^2 + P_1^2 + P_2^2\right],$$

and for the ratio between the elements of area of the pedal and counter-pedal

$$\frac{U}{\bar{V}} = \frac{\sqrt{\{(P^2+u)^2(P^2+v)^2 + P^2P_1^2(P^2+v)^2 + P^2P_2^2(P^2+u)^2\}}}{P^3\sqrt{P^2 + P_1^2 + P_2^2}}.$$

Calling α_2, β_2, γ_2 and $\bar{\alpha}_2$, $\bar{\beta}_2$, $\bar{\gamma}_2$ the direction-cosines of the normals at corresponding points to the counter-pedal and pedal respectively, this ratio is the value of the determinant

$$\begin{vmatrix} \alpha_2 & \frac{dx}{d\bar{x}} & \frac{dx}{d\bar{y}} & \frac{dx}{d\bar{z}} \\ \beta_2 & \frac{dy}{d\bar{x}} & \frac{dy}{d\bar{y}} & \frac{dy}{d\bar{z}} \\ \gamma_2 & \frac{dz}{d\bar{x}} & \frac{dz}{d\bar{y}} & \frac{dz}{d\bar{z}} \\ & \bar{\alpha}_2 & \bar{\beta}_2 & \bar{\gamma}_2 \end{vmatrix}.$$

Let $F(\xi\eta\zeta) = 0$ represent any surface, and denote by F_ξ, F_η, F_ζ the derivatives of F with respect to ξ, η, ζ. The equation of the tangent plane to this surface is (x', y', z' denoting current coördinates in the plane)

$$(x-\xi)F_\xi + (y-\eta)F_\eta + (z-\zeta)F_\zeta = 0;$$

taking the origin as the pole we have for the perpendicular upon the tangent plane

$$p = -\frac{\xi F_\xi + \eta F_\eta + \zeta F_\zeta}{Q},$$

where

$$Q^2 = F_\xi^2 + F_\eta^2 + F_\zeta^2.$$

The direction-cosines of the normal, and therefore of p, are

$$\frac{F_\xi}{Q}, \quad \frac{F_\eta}{Q}, \quad \frac{F_\zeta}{Q};$$

and the coördinates of a point on the pedal surface are

$$\bar{x}, \bar{y}, \bar{z} = \frac{pF_\xi}{Q}, \quad \frac{pF_\eta}{Q}, \quad \frac{pF}{Q}.$$

Writing $\frac{p}{Q} = k$ we have for the coördinates of a point on the counter-pedal

$$\begin{aligned} x &= \xi + kF_\xi, \\ y &= \eta + kF_\eta, \\ z &= \zeta + kF_\zeta. \end{aligned}$$

The elimination of ξ, η, ζ between these equations and $F = 0$ will give the equation of the counter-pedal of $F = 0$.

Write the coördinates x, y, z in the form

$$x = \xi + \bar{x}, \quad y = \eta + \bar{y}, \quad z = \zeta + \bar{z}$$

and we have for $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ the general values

$$\mathfrak{E} = E + \bar{E} + 2\Sigma \frac{d\xi}{du} \frac{d\bar{x}}{du} \text{ \&c.}$$

the quantities $\frac{d\xi}{du}$, $\frac{d\eta}{du}$, $\frac{d\zeta}{du}$ are proportional to the direction-cosines of the tangent to the curve $u = \text{const.}$ traced on the original surface; and $\frac{d\bar{x}}{du}$, $\frac{d\bar{y}}{du}$, $\frac{d\bar{z}}{du}$ are proportional to the direction-cosines of the tangent to the corresponding curve $u = \text{const.}$ traced on the pedal; calling θ the angle between these two tangents the above value of $\mathfrak{E}$ is

$$\mathfrak{E} = E + \bar{E} + 2\sqrt{E\bar{E}} \cos\theta .$$

Writing $\theta_1 = (u, \bar{u})$, we may also write

$$\theta_2 = (v, \bar{v}), \quad \phi_1 = (u, \bar{v}), \quad \phi_2 = (\bar{u}, v).$$

The whole set of values is then

$$\mathfrak{E} = E + \bar{E} + 2\sqrt{E\bar{E}} \cos\theta_1,$$

$$\mathfrak{F} = F + \bar{F} + 2\sqrt{E\bar{G}} \cos\phi_1 + 2\sqrt{\bar{E}G} \cos\phi_2,$$

$$\mathfrak{G} = G + \bar{G} + 2\sqrt{G\bar{G}} \cos\theta_2.$$

It would be interesting to find the surfaces for which the square of the element of length on the counter-pedal is equal to the sum of the squares of the corresponding elements on the given surface and the pedal, *i. e.* the surfaces which give the relation

$$\mathfrak{E}du^2 + 2\mathfrak{F}dudv + \mathfrak{G}dv^2 = Edu^2 + 2Fdudv + Gdv^2 + \bar{E}du^2 + 2\bar{F}dudv + \bar{G}dv^2$$

or

$$d\sigma^2 = ds^2 + d\bar{\sigma}^2.$$

The conditions for this are obviously

$$\cos\theta_1 = 0, \quad \cos\theta_2 = 0$$

and

$$\sqrt{E\bar{G}} \cos\phi_1 + \sqrt{\bar{E}G} \cos\phi_2 = 0;$$

or otherwise

$$\Sigma \frac{d\xi}{du}\frac{d\bar{x}}{du} = 0, \quad \Sigma \frac{d\xi}{dv}\frac{d\bar{x}}{dv} = 0$$

$$\Sigma \frac{d\xi}{du}\frac{d\bar{x}}{dv} + \Sigma \frac{d\xi}{dv}\frac{d\bar{x}}{du} = 0.$$

The quantities $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ may be written in the form

$$\mathfrak{F} = A + \bar{A} - \left\{ \frac{d(\eta, \bar{z})}{d(u, v)} - \frac{d(\zeta, \bar{y})}{d(u, v)} \right\}$$

$$\mathfrak{G} = B + \bar{B} - \left\{ \frac{d(\zeta, \bar{x})}{d(u, v)} - \frac{d(\xi, \bar{z})}{d(u, v)} \right\}$$

$$\mathfrak{H} = C + \bar{C} - \left\{ \frac{d(\xi, \bar{y})}{d(u, v)} - \frac{d(\eta, \bar{x})}{d(u, v)} \right\}.$$

There are many other general formulas that one might write down, but it is hardly worth while to do so.

Tables for the Binary Sextic.

The Leading Coefficients of the First 18 *of the* 26 *Covariants.*

By Professor Cayley.

Including the sextic itself, the number of covariants of the binary sextic is $=26$, as shown in the table p. 296 of Clebsch's "Theorie der binären algebraischen Formen," Leipzig 1872; viz. this is

Deg. \ Order.	0	2	4	6	8	10	12
1				f			
2	A		i		H		
3		l		p	(f, i)		T
4	B		$(f, l)_2$	(f, l)		(H, i)	
5		$(i, l)_2$	(i, l)		(H, l)		
6	A_{ll}			(p, l) $((f, i), l)_2$			
7		$(f, l^2)_4$	$(f, l^2)_3$				
8		$(i, l^2)_3$					
9			$((f, i), l^2)_4$				
10	$(f, l^3)_6$	$(f, l^3)_5$					
12		$((f, i), l^3)_6$					
15	$((f, i), l^4)_8$						

Or, using the capital letters A, B, . . . Z to denote the 26 covariants in the same order, the table is

	0	2	4	6	8	10	12
1				A			
2	B		C		D		
3		E		F	$G, = (A, C)^1$		H
4	I		$J, = (A, E)^2$	$K, = (A, E)^1$		$L, = (D, C)^1$	
5		$M, = (C, E)^2$	$N, = (C, E)^1$		$O, = (D, E)^1$		
6	P			$Q, = (F, E)^1$ $R, = (G, E)^2$			
7		$S, = (A, E^2)^4$	$T, = (A, E^2)^3$				
8		$U, = (C, E^2)^3$					
9			$V, = (G, E^2)^4$				
10	$W, = (A, E^3)^6$	$X, = (A, E^3)^5$					
12		$Y, = (G, E^3)^6$					
15	$Z, = (G, E^4)^8$						

A is the sextic.
B " Salmon's A, p. 202.
I " " B, p. 203.
P is Salmon's C, p. 204.
W " " D, p. 207.
Z " " E, p. 253.
The references are to Salmon's Higher Algebra, 2d Ed., 1866.

In the present short paper I give the leading coefficients of the first 18 covariants, A to R (some of these are of course known values, but it is convenient to include them): for the next four covariants S, T, U, V, the leading coefficients depend upon the coefficients of A, C, G and E^2, viz. writing

$$A = (a,\ b,\ c,\ d,\ e,\ f,\ g \between x,\ y)^6$$
$$E^2 = (\alpha,\ \tfrac{1}{4}\beta,\ \tfrac{1}{6}\gamma,\ \tfrac{1}{4}\delta,\ \varepsilon \between x,\ y)^4$$
$$C = (\alpha',\ \tfrac{1}{4}\beta',\ \tfrac{1}{6}\gamma',\ \tfrac{1}{4}\delta',\ \varepsilon' \between x,\ y)^4$$
$$G = (\alpha'',\ \tfrac{1}{8}\beta'',\ \tfrac{1}{28}\gamma'',\ \tfrac{1}{56}\delta'',\ \tfrac{1}{70}\varepsilon'',\ \ldots \between x,\ y)^8$$

we have

$$\begin{aligned}
&S, \text{ Coeff. } x^2 = a\varepsilon - b\delta + c\gamma - d\beta + e\alpha\,,\\
&T, \quad " \quad x^4 = a\delta - 2b\gamma + 3c\beta - 4d\alpha\,,\\
&U, \quad " \quad x^2 = 2\alpha'\delta - \beta'\gamma + \gamma'\beta - 2\delta'\alpha\,,\\
&V, \quad " \quad x^4 = 280\alpha''\varepsilon - 35\beta''\delta + 10\gamma''\gamma - 20\delta''\beta + 24\varepsilon''\alpha\,.
\end{aligned}$$

Similarly the invariant W and the leading coefficients of X, Y depend on the coefficients of A, G and E^3; and the invariant Z depends on the coefficients of G and E^4. But these two invariants W and Z have been already calculated; viz., as already mentioned, W is Salmon's invariant D, and Z his invariant E, given each of them in the second edition of his Higher Algebra (but not reproduced in the third edition): on account of the great length of these expressions it has been thought that it was not expedient to give them here.

For the reason appearing above, I have added the expressions for the remaining coefficients of C, E, G.

A, x^6	B, x^0	C, x^4	D, x^8	E, x^2	F, x^6	G, x^8	H, x^{12}
$a + 1$	$ag + 1$	$ae + 1$	$ac + 1$	$acg + 1$	$ace + 1$	$a^2f + 1$	$a^2d + 1$
	$a^0bf - 6$	$a^0bd - 4$	$a^0b^2 - 1$	$df - 3$	$d^2 - 1$	$abe - 5$	$abc - 3$
	$ce + 15$	$c^2 + 3$		$e^2 + 2$	$a^0b^2e - 1$	$cd + 2$	$a^0b^3 + 2$
	$d^2 - 10$			$a^0b^2g - 1$	$bcd + 2$	$a^0b^2d + 8$	
				$bcf + 3$	$c^3 - 1$	$bc^2 - 6$	
				$bde - 1$			
				$c^2e - 3$			
				$cd^2 + 2$			

$I,\ x^0$		$J,\ x^4$		$K,\ x^6$		$L,\ x^{10}$		$M,\ x^2$		$N,\ x^4$		$O,\ x^8$	
$aceg$	$+1$	a^2f^2	$+1$	a^2dg	$+1$	a^2cf	$+1$	a^2cg^2	$+1$	a^2cfg	-1	a^2cdg	0
cf^2	-1	$abef$	-10	ef	-1	de	-1	dfg	-6	deg	$+1$	cef	-1
d^2g	-1	cdf	$+4$	$abcg$	-3	ab^2f	-1	e^2g	$+8$	df^2	$+3$	d^2f	$+3$
def	$+2$	ce^2	$+16$	bdf	-2	bce	-2	ef^2	-3	e^2f	-3	de^2	-2
e^3	-1	d^2e	-12	be^2	$+5$	bd^2	$+4$	ab^2g^2	-1	ab^2fg	$+1$	ab^2dg	0
a^0b^2eg	-1	a^0b^2df	$+16$	c^2f	$+9$	c^2d	-1	$bcfg$	$+6$	$bceg$	$+2$	b^2ef	$+1$
b^2f^2	$+1$	b^2e^2	$+9$	cde	-17	a^0b^3e	$+3$	$bdeg$	-34	bcf^2	-3	bc^2g	0
$bcdg$	$+2$	bc^2f	-12	d^3	$+8$	b^2cd	-6	bdf^2	$+48$	bd^2g	-4	$bcdf$	-14
$bcef$	-2	$bcde$	-76	a^0b^3g	$+2$	bc^3	$+3$	be^2f	-18	$bdef$	-12	bce^2	$+11$
bd^2f	-2	bd^3	$+48$	b^2cf	-6			c^2eg	$+18$	be^3	$+15$	bd^2e	$+1$
bde^2	$+2$	c^3e	$+48$	b^2de	$+2$			c^2f^2	-45	c^2dg	$+1$	c^3f	$+9$
c^3g	-1	c^2d^2	-32	bc^2e	$+6$			cd^2g	$+4$	c^2ef	$+9$	c^2de	-14
c^2df	$+2$			bcd^2	-4			$cdef$	$+78$	cd^2f	$+4$	cd^3	$+6$
c^2e^2	$+1$							ce^3	-36	cde^2	-21	a^0b^3cg	0
cd^2e	-3							d^3f	-48	d^3e	$+8$	b^3df	$+8$
d^4	$+1$							d^2e^2	$+28$	a^0b^3eg	-3	b^3e^2	-9
								a^0b^2ceg	0	b^2cdg	$+6$	b^2c^2f	-6
								b^2d^2g	$+64$	b^2cef	$+9$	b^2cde	$+16$
								b^2def	-144	b^2d^2f	$+32$	b^2d^3	-8
								b^2e^3	$+81$	b^2de^2	-39	bc^3e	-3
								bc^2dg	-96	bc^3g	-3	bc^2d^2	$+2$
								bc^2ef	$+108$	bc^2df	-66		
								bcd^2f	$+96$	bc^2e^2	$+18$		
								$bcde^2$	-126	bcd^2e	$+76$		
								bd^3e	$+16$	bd^4	-32		
								c^4g	$+36$	c^4f	$+27$		
								c^3df	-72	c^3de	-45		
								c^3e^2	-27	c^2d^3	$+20$		
								c^2d^2e	$+96$				
								cd^4	-32				

P, x^0

$a^2d^2g^2$	+	1
$defg$	−	6
df^3	+	4
e^3g	+	4
e^2f^2	−	3
$abcdg^2$	−	6
$bcefg$	+	18
bcf^3	−	12
bd^2fg	+	12
bde^2g	−	18
be^3f	+	6
c^3g^2	+	4
c^2e^2g	−	24
c^2dfg	−	18
c^2ef^2	+	30
cd^2eg	+	54
cd^2f^2	−	12
cde^2f	−	42
ce^4	+	12
d^4g	−	20
d^3ef	+	24
d^2e^3	−	8
$a^0b^3dg^2$	+	4
b^3efg	−	12
b^3f^3	+	8
$b^2c^2g^2$	−	3
b^2ce^2g	+	30
b^2cef^2	−	24
b^2d^2eg	−	12
$b^2d^2f^2$	−	24
b^2de^2f	+	60
b^2e^4	−	27
bc^3fg	+	6
bc^2deg	−	42
bc^2df^2	+	60
bc^2e^2f	−	30
bcd^3g	+	24
bcd^2ef	−	84
$bcde^3$	+	66
bd^4f	+	24
bd^3e^2	−	24
c^4eg	+	12
c^4f^2	−	27
c^3d^2g	−	8
c^3def	+	66
c^3e^3	−	8
c^2d^3f	−	24
$c^2d^2e^2$	−	39
cd^4e	+	36
d^6	−	8

	Q, x^6		R, x^6	
a^3dg^2			−	1
efg			+	9
f^3			−	8
a^2bcg^2			+	3
$bdfg$			−	24
be^2g			−	45
bef^2			+	66
c^2fg	−	2	+	3
$cdeg$	+	5	+	48
cdf^2	+	6	−	12
ce^2f	−	7	−	51
d^3g	−	3	−	16
d^2ef	−	3	+	36
de^3	+	4	−	8
ab^3g^2		0	−	2
b^2cfg	+	4	+	12
b^2deg	−	5	+	192
b^2df^2	−	6	−	48
b^2e^2f	+	7	−	144
bc^2eg	−	5	−	159
bc^2f^2	−	6	+	18
bcd^2g	+	7	−	48
$bcdef$	−	16	+	24
bce^3	+	23	+	279
bd^3f	+	30	−	48
bd^2e^2	−	33	−	84
c^3dg	−	1	+	42
c^3ef	+	36	+	153
c^2d^2f	−	37	−	36
c^2de^2	−	53	−	399
cd^3e	+	79	+	312
d^5	−	24	−	64
a^0b^4fg	−	2		0
b^3ceg	+	5		0
b^3cf^2	+	6		0
b^3d^2g	+	2	−	224
b^3def	+	22	+	144
b^3e^3	−	27	+	54
b^2c^2dg	−	8	+	336
b^2c^2ef	−	39	−	108
b^2cd^2f	−	50	+	384
b^2cde^2	+	107	−	684
b^2d^3e	−	22	+	144
bc^4g	+	3	−	126
bc^3df	+	84	−	648
bc^3e^2	−	21	+	432
bc^2d^2e	−	102	+	564
bcd^4	+	44	−	288
c^5f	−	27	+	270
c^4de	+	45	−	450
c^3d^3	−	20	+	200

Remaining Coefficients of C, E, G.

C

x^3y	
af	$+2$
be	-6
cd	$+4$
x^2y^2	
ag	$+1$
ce	-9
d^2	$+8$
xy^3	
bg	$+2$
cf	-6
de	$+4$
y^4	
cg	$+1$
df	-4
e^2	$+3$

E

xy	
adg	$+1$
aef	-1
bcg	-1
bdf	-8
be^2	$+9$
c^2f	$+9$
cde	-17
d^3	$+8$
y^2	
aeg	$+1$
af^2	-1
bdg	-3
bef	$+3$
c^2g	$+2$
cdf	-1
ce^2	-3
d^2e	$+2$

G

x^7y	
a^2g	$+1$
abf	$+2$
ace	-19
ad^2	$+8$
b^2e	-6
bcd	$+44$
c^3	-30
x^6y^2	
abg	$+7$
acf	-14
ade	-14
b^2f	0
bce	-21
bd^2	$+112$
c^2d	-70
x^5y^3	
acg	$+7$
adf	-28
ae^2	-14
b^2g	$+14$
bcf	-42
bde	$+168$
c^2e	-105
x^4y^4	
adg	0
aef	-35
bcg	$+35$
bdf	0
be^2	$+105$
c^2f	-105

G

x^3y^5	
aeg	-7
af^2	-14
bdg	$+28$
bef	$+42$
c^2g	$+14$
cdf	-168
ce^2	$+105$
x^2y^6	
afg	-7
beg	$+14$
bf^2	0
cdg	$+14$
cef	$+21$
d^2f	-112
de^2	$+70$
xy^7	
ag^2	-1
bfg	-2
ceg	$+19$
cf^2	$+6$
d^2g	-8
def	-44
e^3	$+30$
y^8	
bg^2	-1
cfg	$+5$
deg	-2
df^2	-8
e^2f	$+6$

Note.—In the tables on this page, a has been treated like the other letters; on the preceding pages, the powers of a have been suppressed except in the first of every series of terms containing a common power of a.

The final result is that we have the values of the invariants B, I, P, W, Z and the leading coefficients of the covariants A, C, D, E, F, G, H, J, K, L, M, N, O, Q, R: also the means of calculating the leading coefficients of the remaining covariants S, T, U, V, X, Y.

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